BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(44), 2004, Pages 76–84
ISSN 1024–7696

Radicals around Köthe's problem

S. Tumurbat, R. Wiegandt

Abstract. Radicals γ will be studied for which the condition "$A[x] \in \gamma$ for all nil rings A" is equivalent to the positive solution of Köthe's Problem ($A[x]$ is Jacobson radical for all nil rings A, in Krempa's formulation). The closer γ is to the Jacobson radical, the better approximation of the positive solution is obtained. Seeking, however, for a negative solution, possibly large radicals γ are of interest. In this note such large radicals will be studied.

Mathematics subject classification: Primary: 16N80. Secondary: 16N20, 16N40.
Keywords and phrases: Primitive and polynomial ring, Köthe's Problem, nil and Jacobson radical.

1 Introduction

We shall work with associative rings (not necessarily with unity element) and Kurosh–Amitsur radicals. For details we refer to [4]. We shall use the following letters for operators acting on classes of rings:

$\mathcal{L}$ lower radical operator;
$\mathcal{U}$ upper radical operator;
h homomorphic closure operator;
$\mathcal{H}$ hereditary closure operator.

Further notations:

$\mathcal{N} = \{\text{all nil rings}\}$, the nil radical class;
$\mathcal{J}$ the Jacobson radical or radical class;
$\mathcal{G}$ the Brown–McCoy radical or radical class;
$\mathcal{B}$ the Behrens radical: the upper radical of rings with nonzero idempotents;
u the upper radical of uniformly strongly prime rings (a ring A is uniformly strongly prime, if there exists a finite subset $F \subseteq A$ such that $xFy \neq 0$ whenever $0 \neq x, y \in A$);
$\mathcal{P} = \{\text{all primitive rings}\}$;
$\mathcal{Q} = \{A[x] \mid A \in \mathcal{N}\}$;
$\ell = \mathcal{L}h\mathcal{Q}$;
$\mathfrak{K} = \mathcal{U}\mathcal{H}(\ell \cap \mathcal{P})$;
$\mathfrak{M} = \mathcal{U}(\ell \cap \mathcal{P})$ may not be a radical class, though homomorphically closed.

Research supported by the Hungarian OTKA Grants T034530 and T043034.

Köthe's Problem (1930) asks as whether the sum of two nil left ideals is always a nil left ideal. Krempa's well-known criterion says that Köthe's Problem is equivalent to the condition: $A[x] \in \mathcal{J}$ for every $A \in \mathcal{N}$, that is, $\mathcal{Q} \subset \mathcal{J}$.

This raises the possibility of approximating Köthe's Problem by radicals. At present, by [1], [2] we know that $A \in \mathcal{N}$ implies $A[x] \in \mathcal{B} \cap u$. Tumurbat [9] gave the exact lower bound ℓ, and a positive solution of Köthe's Problem is equivalent to $\ell(A[x]) \subseteq \mathcal{J}(A[x])$ for all nil rings A.

McConnell and Stokes [6] introduced and investigated a non-hereditary radical $\mathcal{K}$, they proved that $\mathcal{J} \subset \mathcal{K}$ and that Köthe's Problem has a positive solution if and only if $A[x] \in \mathcal{K}$ for every $A \in \mathcal{N}$. Recently Sakhajev [7] announced the negative solution of Köthe's Problem. Thus, solving Köthe's problem in the negative by an explicitly given counterexample, possibly large radicals γ may be of interest for which $\mathcal{J} \subset \gamma$ and $A[x] \in \gamma$ for every $A \in \mathcal{N}$. In this note we shall investigate such large radicals.

2 An interval of radicals

Proposition 2.1. (i) *For a radical γ, $\mathcal{J} \cap \mathcal{Q} = \gamma \cap \mathcal{Q}$ if and only if $\mathcal{J} \cap h\mathcal{Q} = \gamma \cap h\mathcal{Q}$;*

(ii) *$A \in \mathcal{N}$ and $A[x] \in \mathfrak{M}$ implies $A[x] \in \mathcal{J} \cap \ell$;*

(iii) *$\mathfrak{M} \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q} = (\mathcal{J} \cap \ell) \cap \mathcal{Q}$.*

Proof. (i) Straightforward.

(ii) If $A \in \mathcal{N}$ and $A[x] \notin \mathcal{J}$, then $A[x]$ has a nonzero homomorphic image in $\ell \cap \mathcal{P}$; so $A[x] \notin \mathfrak{M}$. Hence $A \in \mathcal{N}$ and $A[x] \in \mathfrak{M}$ implies $A[x] \in \mathcal{J}$, whence $A[x] \in \mathcal{J} \cap \ell$.

(iii) Obvious by (ii). □

Proposition 2.2. *Let γ be any radical. Then*

(i) *$\gamma \in [\ell \cap \mathcal{J}, \mathfrak{M}]$ implies $\gamma \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}$;*

(ii) *$\gamma \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}$ implies $\ell \cap \mathcal{J} \subseteq \gamma$;*

(iii) *if γ is hereditary and $\gamma \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}$, then $\gamma \subseteq \mathfrak{M}$.*

Proof. (i) Since $\mathcal{Q} \subset \ell$, the equality $(\ell \cap \mathcal{J}) \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}$ is obvious.

Next, we prove that $\mathfrak{M} \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}$. Clearly $\mathcal{J} \subseteq \mathfrak{M}$, therefore $\mathcal{J} \cap \mathcal{Q} \subseteq \mathfrak{M} \cap \mathcal{Q}$. Assume that there exists a ring $A[x] \in (\mathfrak{M} \cap \mathcal{Q}) \setminus \mathcal{J}$. Then $A[x] \in \ell$ and has a nonzero homomorphic image B in $\mathcal{P}$. Hence $B \in \ell \cap \mathcal{P}$, and so $A[x] \notin \mathfrak{M}$. This contradiction proves that $\mathfrak{M} \cap \mathcal{Q} \subseteq \mathcal{J} \cap \mathcal{Q}$.

Let γ be any radical class in the interval $[\ell \cap \mathcal{J}, \mathfrak{M}]$. Then we have

$$\mathcal{J} \cap \mathcal{Q} = (\ell \cap \mathcal{J}) \cap \mathcal{Q} \subseteq \gamma \cap \mathcal{Q} \subseteq \mathfrak{M} \cap \mathcal{Q} = \mathcal{J} \cap \mathcal{Q}.$$

(ii) Assume that $\ell \cap \mathcal{J} \nsubseteq \gamma$, and $A \in (\ell \cap \mathcal{J}) \setminus \gamma$. Then every nonzero homomorphic image B of A has a nonzero accessible subring C in $h\mathcal{Q} \cap \mathcal{J} = h\mathcal{Q} \cap \gamma \subseteq \gamma$. Hence $A \in \gamma$ follows, contradicting $A \notin \gamma$.

(iii) Suppose that $\gamma \nsubseteq \mathfrak{M}$ and $A \in \gamma \setminus \mathfrak{M}$. Then A has a nonzero homomorphic image $B \in \gamma \cap \ell \cap \mathcal{P}$, and therefore B has a nonzero accessible subring $C \in h\mathcal{Q} \cap \mathcal{P}$.

Since γ is hereditary, also $C \in \gamma \cap h\mathcal{Q} \cap \mathcal{P}$ holds. Hence $\gamma \cap h\mathcal{Q} \nsubseteq \mathcal{J} \cap h\mathcal{Q}$ follows. In view of Proposition 2.1 (i) this is a contradiction. □

Theorem 2.3. *A radical γ is in the interval $[\ell \cap \mathcal{J}, \mathfrak{M}]$ if and only if $\gamma \cap \ell = \mathcal{J} \cap \ell$.*

Proof. Assume that $\gamma \in [\ell \cap \mathcal{J}, \mathfrak{M}]$. Clearly $\ell \cap \mathcal{J} \subseteq \gamma$ and so $\ell \cap \mathcal{J} \subseteq \ell \cap \gamma$. Suppose that $\ell \cap \mathcal{J} \neq \ell \cap \gamma$. Then there exists a ring $A \in (\ell \cap \gamma) \setminus (\ell \cap \mathcal{J})$, and necessarily $A \notin \mathcal{J}$. Hence A has a nonzero homomorphic image $B \in \mathcal{P} \cap \ell$, and so $A \notin \mathfrak{M}$, contradicting $A \in \ell \cap \mathcal{J} \subseteq \gamma \subseteq \mathfrak{M}$.

Conversely, suppose that $\gamma \cap \ell = \mathcal{J} \cap \ell$ for some radical γ. We claim that $\gamma \subseteq \mathfrak{M}$. Assume that this is not true, and $\gamma \nsubseteq \mathfrak{M}$. Then there exists a nonzero homomorphic image of a ring $A \in \gamma$ such that $B \in \gamma \cap (\ell \cap \mathcal{P}) = (\mathcal{J} \cap \ell) \cap \mathcal{P} = \{0\}$, a contradiction. Hence $\gamma \subseteq \mathfrak{M}$. Further, $\mathcal{J} \cap \ell = \gamma \cap \ell \subseteq \gamma$. □

Next, we give conditions equivalent to the positive solution of Köthe's problem.

Theorem 2.4. *The following conditions are equivalent:*

(i) *Köthe's problem has a positive solution;*
(ii) $\ell \subseteq \mathcal{J}$*;*
(iii) $\ell \cap \mathcal{P} = \{0\}$*;*
(iv) $\mathfrak{K} = \{$*all rings*$\} = \mathfrak{M}$*;*
(v) $\mathcal{Q} \subseteq \gamma$ *for any radical* γ *with* $\ell \cap \mathcal{J} \subseteq \gamma$*.*

Proof. The following implications are straightforward:

(i)$\Longrightarrow$(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv),
(ii)$\Longrightarrow$(v)$\Longrightarrow$(i). □

Corollary 2.5. *Let γ be a radical such that $\mathcal{Q} \subset \gamma$.*

Then $\gamma \subseteq \mathfrak{M}$ if and only if Köthe's problem has a positive solution. In particular, γ may be the Behrens, Brown–McCoy, uniformly strongly prime radicals, or the upper radical of von Neumann regular rings.

Proof. $\mathcal{Q} \subset \gamma$ implies $\ell \subseteq \gamma$, and so $\ell \cap \mathcal{J} \subseteq \gamma$. Hence from Theorem 2.3 it follows that $\ell \subseteq \gamma \cap \ell = \mathcal{J} \cap \ell \subseteq \mathcal{J}$. Further, $\mathcal{Q} \subset \mathcal{B} \cap \mathcal{G} \cap u \cap \mathcal{U}\nu$ is well-known (see [1] and [2]). □

A radical γ is said to be *polynomially extensible* if $A \in \gamma$ implies $A[x] \in \gamma$.

Corollary 2.6. *Köthe's problem has a positive solution if and only if the interval $[\ell \cap \mathcal{J}, \mathfrak{M}]$ contains a polynomially extensible radical.*

Proof. If Köthe's problem has a positive solution, then $\mathfrak{M} = \{\text{all rings}\}$ is polynomially extensible.

Let $\gamma \in [\ell \cap \mathcal{J}, \mathfrak{M}]$ be a polynomially extensible radical. Then $\mathcal{N} \subseteq \ell \cap \mathcal{J} \subseteq \gamma$ implies $\mathcal{Q} \subseteq \gamma$. Hence Theorem 2.4 (v) yields the assertion. □

Remark 2.7. *In Corollaries 2.5 and 2.6 the class $\mathfrak{M}$ can be replaced by the radical $\mathfrak{K}$.*

3 On the radical $\mathfrak{K}$

As we have seen in Theorem 2.4, the radical $\mathfrak{K}$ is not the class of all rings if and only if $\ell \cap \mathcal{P} \neq \{0\}$. This is the case precisely when there exists a polynomial ring $R[x]$ over a nil ring R which has a nonzero primitive homomorphic image. In this section we shall discuss properties of the radical $\mathfrak{K}$ and prove criteria of the positive solution of Köthe's Problem in terms of $\mathfrak{K}$.

Theorem 3.1. *Köthe's Problem has a positive solution if and only if the radical $\mathfrak{K}$ is hereditary.*

Proof. If Köthe's Problem has a positive solution, then $\ell \cap \mathcal{P} = \{0\}$ and $\mathfrak{K}$ is the class of all rings, which is trivially hereditary.

Conversely, suppose that $\mathfrak{K}$ is hereditary. Let us consider an arbitrary nonzero ring A and its Dorroh extension A^1. We are going to prove that $A^1 \in \mathfrak{K}$. Suppose the contrary, that $A^1 \notin \mathfrak{K}$. Then A^1 has a nonzero homomorphic image $B^1 \in \mathcal{H}(\ell \cap \mathcal{P})$. Hence B^1 is an accessible subring of a ring $C \in \ell \cap \mathcal{P}$, and so has a nonzero idempotent, the unity element e of B^1. By a Zorn lemma argument C has a subdirectly irreducible homomorphic image $C/M \in \ell$ possessing a nonzero idempotent $e + M$ in its heart. Thus C/M is in the Behrens semisimple class $\mathcal{SB}$. Taking into account that $C/M \in \ell$, we conclude that there exists a nonzero accessible subring D of C/M which is in $\mathcal{SB} \cap h\mathcal{Q}$. Thus there exists a polynomial ring $E[x]$ over a nil ring E such that $D \cong E[x]/K$. But by Beidar, Fong and Puczyłowski [1], $E \in \mathcal{N}$ implies $E[x] \in \mathcal{B}$ and also $D \in \mathcal{B}$, a contradiction. Hence $A^1 \in \mathfrak{K}$. Thus by $A \triangleleft A^1$ the hereditariness of $\mathfrak{K}$ yields $A \in \mathfrak{K}$, which means that $\mathfrak{K}$ is the class of all rings, and so $\ell \cap \mathcal{P} = \{0\}$ and $\ell \subseteq \mathcal{J}$ follows. Hence $A \in \mathcal{N}$ implies $A[x] \in \mathcal{J}$. $\square$

A ring A is said to be an *s-ring* if every primitive homomorphic image of A is a reduced ring or has a homomorphic image with nonzero idempotent. Recall that the class $\mathcal{L}$ of locally nilpotent rings is the *Levitzki radical class*. In the proof of the next Proposition and in Theorem 4.5 we shall make use of the radicals ϱ and δ which are *the upper radicals of the classes*

$$\{A \in \mathcal{SL} \mid \text{every nil subring of } A \text{ is in } \mathcal{L}\}$$

and

$$\{A \in \mathcal{SN} \mid \text{the nilpotent elements of } A \text{ form a subring}\},$$

respectively.

Proposition 3.2. *All s-rings are in the radical class $\mathfrak{K}$.*

Proof. Suppose that A is an s-ring and $A \notin \mathfrak{K}$. Then A has a nonzero homomorphic image B which is an accessible subring of a ring $C \in \ell \cap \mathcal{P}$. Hence also B is a primitive ring. Suppose that B is a reduced ring. We choose an ideal I of C which is maximal relative to $I \cap B = 0$. Then by $B \cong (B + I)/I$, we may assume that B is an accessible subring in $D = C/I$. By induction we can see that B is an essential accessible subring in D. Since B is primitive, we conclude that also D is primitive.

By an iterated application of the Andrunakievich Lemma we get that a power J of the ideal of D generated by B, is contained in B. Since B is primitive, necessarily $J \neq 0$. Thus D has a nonzero ideal J contained in B. J is a reduced ring as B is so. We show that also D is a reduced ring. Assume that $a^2 = 0$ for a nonzero element $a \in D$. Since D is primitive, necessarily $aJa \neq 0$ and so $0 \neq aja \in J$ with a suitable element $j \in J$. Hence $0 = aja \cdot aja \in J$ follows, a contradiction. Thus D is a reduced ring and $D \in \ell$ by $C \in \ell \cap \mathcal{P}$. As proved in [9, Theorem 2.9], $\ell \subset \varrho \cap \delta$, and so $D \in \ell \subset \varrho$. Hence D has a nonzero locally nilpotent ideal or D has a nil subring which is not locally nilpotent, whence D is not reduced, a contradiction. Thus B is not reduced, but B has a homomorphic image possessing a nonzero idempotent as well as a subdirectly irreducible homomorphic image B/K which has a nonzero idempotent in its heart H/K. Let us consider the ideal $\langle K \rangle_D$ of D generated by K. Now we have $K \subseteq \langle K \rangle_D \cap H$. By the simplicity of H/K either $\langle K \rangle_D \cap H = H$ or $\langle K \rangle_D \cap H = K$. In the first case there exists a natural number $n \geq 3$ such that $H = H^n = \langle K \rangle_D^n \cap H \subseteq \langle K \rangle_D^n \subseteq K$, contradicting $H/K \neq 0$. So $\langle K \rangle_D \cap H = K$. Using the Zorn Lemma there exists an ideal M of D which is maximal relative to $M \cap H = K$. Then the factor ring D/M is subdirectly irreducible with heart $(H+M)/M \cong H/K$. Hence $D/M \in \ell \setminus \mathcal{B}$, contradicting $\ell \subset \mathcal{B}$ (cf. [1]). Thus $A \in \mathfrak{K}$ has been established. $\square$

Applying Proposition 3.2 to some special cases of s-rings, we get

Corollary 3.3. *All rings with unity element, all commutative rings and all rings with d.c.c. on principal left ideals are in* $\mathfrak{K}$.

For a ring A we denote by $[A, A]$ the ideal of A generated by the commutators $[a, b] = ab - ba$ for all $a, b \in A$.

Theorem 3.4. *The following conditions are equivalent:*

(i) *Köthe's Problem has a positive solution,*

(ii) *if a finitely generated Jacobson semisimple ring A is in $\mathfrak{K}$, then also its commutator ideal $[A, A]$ is in $\mathfrak{K}$.*

Proof. (i)$\Longrightarrow$(ii) Trivial by Theorem 2.4 (iv).

(ii)$\Longrightarrow$(i) Let F be a finitely generated free ring.

Clearly, also the unital extension F^1 of F is finitely generated, and both F and F^1 are Jacobson semisimple. Hence by Corolary 3.3 the ring F^1 is in $\mathfrak{K}$, and by (ii) we have that $[F^1, F^1] \in \mathfrak{K}$. Again by Corolary 3.3 the commutative ring $F/[F, F]$ is in $\mathfrak{K}$. But $[F, F] = [F^1, F^1] \in \mathfrak{K}$, so also $F \in \mathfrak{K}$.

Suppose that Köthe's Problem has a negative solution. Then there exists a nil ring A such that $A[x] \notin \mathcal{J}$. Hence there exists a polynomial $f(x) = \sum\limits_{i=0}^{n} a_i x^i \in A[x]$ such that $f(x)$ has no quasi-inverse in $A[x]$. Let B denote the subring of A generated by the elements $a_0, \ldots, a_n \in A$. By $B \subseteq A$, also B is a nil ring, further, also the ring C generated by B and x is finitely generated. Since every finitely generated free ring is in $\mathfrak{K}$, we have that $C \in \mathfrak{K}$. Further, from

$$C/B[x] \cong \{x\} \in \mathcal{SJ}$$

it follows that $\mathcal{J}(C) \subseteq \mathcal{J}(B[x])$, and from $B[x] \lhd C$ we conclude that $\mathcal{J}(B[x]) \subseteq \mathcal{J}(C)$. So $D = C/\mathcal{J}(B[x])$ is a finitely generated Jacobson semisimple ring in $\mathfrak{K}$. Applying condition (ii) we get that $[D, D] \in \mathfrak{K}$. Obviously we have

$$[D, D] = [B[x]/\mathcal{J}(B[x]), B[x]/\mathcal{J}(B[x])]\,.$$

Thus, we infer from Corollary 3.3 that

$$\frac{B[x]/\mathcal{J}(B[x])}{[D, D]} \in \mathfrak{K}.$$

Hence $B[x]/\mathcal{J}(B[x]) \in \mathfrak{K}$ and by $\mathcal{J}(B[x]) \in \mathfrak{K}$ also $B[x] \in \mathfrak{K}$ follows. Moreover, using the fact that the Jacobson radical has the Amitsur property, we have

$$\left(\frac{B}{B \cap \mathcal{J}(B[x])}\right)[x] \cong \frac{B[x]}{(B \cap \mathcal{J}(B[x])[x]} = \frac{B[x]}{\mathcal{J}(B[x])} \lhd \frac{C}{\mathcal{J}(B[x])} \in \mathcal{SJ},$$

and so $(B/(B \cap \mathcal{J}(B[x]))[x] \in \mathcal{SJ}$. Thus, taking into account that $B \in \mathcal{N}$, there exists a nonzero homomorphic image E of $(B/B \cap \mathcal{J}(B[x]))[x]$ such that
$E \in \mathcal{P} \cap h\mathcal{Q} \subseteq \mathcal{P} \cap \ell$. Hence $E \notin \mathfrak{K}$, contradicting $B[x] \in \mathfrak{K}$. □

A finitely generated nil ring L is said to be *strongly nil*, if

i) L can be embedded into a ring A as a left ideal,

ii) $A = L + K$ where K is a finitely generated nil left ideal of A and $L \cap K = 0$,

iii) A is generated by two nilpotent elements $x \in L$ and $y \in K$.

Theorem 3.5. *Köthe's Problem has a positive solution if and only if $L[x] \in \mathfrak{K}$ for every strongly nil ring L.*

Proof. Suppose that $L[x] \in \mathfrak{K}$ for every strongly nil ring L, but Köthe's problem has a negative solution. Then, as is well-known (cf. Krempa [5] and Sands [8]), there exists a nil ring B such that the 2×2 matrix ring $M_2(B)$ is not nil. Hence there exists an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(B)$ which is not nilpotent. Nevertheless, the elements $x = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ are nilpotent, as one readily sees. Let A denote the subring of $M_2(B)$ generated by x and y, and L and K the left ideals of A generated by x and y, respectively. Obviously, L as a ring is generated by elements $y^i x^k$ ($i \geq 0$, $k \geq 1$) and K is generated by $x^k y^i$ ($k \geq 0$, $i \geq 1$). Since x and y are nilpotent elements, both L and K are finitely generated. Clearly $A = L + K$ and $L \cap K = 0$. Thus both L and K are strongly nil rings, and so by the assumption $L[x] \in \mathfrak{K}$ and $K[x] \in \mathfrak{K}$. Hence by Proposition 2.1 (ii) we have $L[x] \in \mathcal{J}$, $K[x] \in \mathcal{J}$, and therefore $A[x] = (L + K)[x] = L[x] + K[x] \in \mathcal{J}$. But then A is a nil ring, a contradiction.

The opposite implication is obvious. □

4 Radicals in $[\mathcal{J}, \mathfrak{M}]$

Seeking for a positive solution of Köthe's Problem, it is of interest to find radicals γ for which $A \in \mathcal{N}$ implies $A[x] \in \gamma$ and γ is "close" to $\mathcal{J}$. Sakhajev [7], however, states that Köthe's Problem has a negative solution. Searching for a counterexample, one has to find a nil ring A such that $A[x] \notin \gamma$ where γ is a possibly large radical in the interval $[\mathcal{J}, \mathfrak{M}]$. The main goal of this section is to construct a rather big radical Ξ such that $\mathcal{J} \subset \Xi \subset \mathfrak{M}$.

McConnell and Stokes [6] considered the following generalization of the Jacobson radical class

$$\mathcal{K} = \{A \mid (A, \circ) \text{ is a simple semigroup}\}$$

where $\circ$ denotes the adjoint operation $a \circ b = a + b + ab$ and simplicity means that the semigroup has no proper ideals. In [6] it was proved, among others, that

(1) *$\mathcal{K}$ is a non-hereditary radical;*

(2) *$\mathcal{J} \subset \mathcal{K} \subset \mathcal{G}$ and $\mathcal{J} = \mathcal{K} \cap \mathcal{B}$;*

(3) *Köthe's problem has a positive solution if and only if $A \in \mathcal{N}$ implies $A[x] \in \mathcal{K}$.*

Notice that by (2), (3) requires seemingly less than Krempa's criterion $A \in \mathcal{N} \Rightarrow A[x] \in \mathcal{J}$. Moreover, looking at the original definition of $\mathcal{K}$ in [6], one sees that $\mathcal{K}$ *is a polynomial but not a multiplicative radical in the sense of Drazin and Roberts* [3].

Proposition 4.1. $\mathcal{K} \in [\ell \cap \mathcal{J}, \mathfrak{K}]$.

Proof. By (2) the containment $\ell \cap \mathcal{J} \subset \mathcal{K}$ is clear. We show that $\mathcal{J} \cap \mathcal{Q} = \mathcal{K} \cap \mathcal{Q}$. Let $A[x] \in \mathcal{K} \cap \mathcal{Q}$. Then by [1] we have $A[x] \in \mathcal{B}$, and so $A[x] \in \mathcal{K} \cap \mathcal{B} = \mathcal{J}$. Thus $\mathcal{K} \cap \mathcal{Q} \subseteq \mathcal{J} \cap \mathcal{Q}$. The opposite inclusion is trivial. Applying Proposition 2.2 (iii), we get that $\mathcal{K} \subseteq \mathfrak{M}$ and also $\mathcal{K} \subseteq \mathfrak{K}$. $\square$

For a radical γ we consider the classes

$$\mu_\gamma = \{A \in \mathcal{S}\gamma \mid A \text{ is a prime ring with a minimal left ideal}\}$$

and

$$\nu_\gamma = \left\{ A \in \mathcal{S}\gamma \,\middle|\, \begin{array}{l} \text{every nonzero prime homomorphic image of } A \\ \text{which is in } \mathcal{S}\gamma\text{, has no minimal left ideals} \end{array} \right\}.$$

Proposition 4.2. *If γ is a special radical, then $\gamma = \mathcal{U}(\mu_\gamma \cup \nu_\gamma) = \mathcal{U}\mu_\gamma \cap \mathcal{U}\nu_\gamma$.*

Proof. The inclusion $\gamma \subseteq \mathcal{U}(\mu_\gamma \cup \nu_\gamma)$ is obvious. For proving $\mathcal{U}(\mu_\gamma \cup \nu_\gamma) \subseteq \gamma$, suppose the contrary. Then there exists a ring $A \in \mathcal{U}(\mu_\gamma \cup \nu_\gamma) \setminus \gamma$. Since γ is a special radical, A has a nonzero prime homomorphic image $B \in \mathcal{S}\gamma$. Certainly $B \notin \nu_\gamma$. Hence B has a nonzero prime homomorphic image C in $\mathcal{S}\gamma$ which has a minimal left ideal. Thus $C \in \mu_\gamma$, a contradiction.

The proof of $\mathcal{U}(\mu_\gamma \cup \mathcal{U}\nu_\gamma) = \mathcal{U}\mu_\gamma \cap \mathcal{U}\nu_\gamma$ is straightforward. $\square$

Let m stand for the class of all subdirectly irreducible rings with minimal left ideals. Then in view of Proposition 4.2 the heart $H(A)$ of any $A \in m$ has a minimal left ideal, and so by the Litoff Theorem $H(A)$ is a locally matrix ring. Hence $H(A)$ contains a nonzero idempotent for every $A \in m$. Moreover the Weyl algebra

$W = \mathbb{Q}\langle x, y\rangle$ of rational polynomials with non-commuting indeterminates subject to $xy - yx = 1$ is a simple ring with unity element which does not contain minimal left ideals. From these considerations we conclude that $\mathcal{B} \subset \mathcal{U}m$.

Let ϱ and δ stand for the radicals introduced before Proposition 3.2, and put $\kappa = \mathcal{B} \cap u \cap \varrho \cap \delta$.

Proposition 4.3. ([9, Theorem 2.9]) *If $A \in \mathcal{N}$, then $A[x] \in \kappa$.*

We shall denote by Ξ the upper radical class of the class

$$\pi = \{A \mid A \text{ is an accessible subring of a primitive ring in } \kappa\}.$$

Proposition 4.4. *$\mathcal{K} \subset \Xi \not\subseteq \mathcal{U}\{S\}$ for every simple ring S with unity element.*

Proof. Suppose that $\mathcal{K} \not\subseteq \Xi$ and there exists a ring $A \in \mathcal{K} \backslash \Xi$. Then A has a nonzero primitive homomorphic image B in $\pi \cap \mathcal{K}$. Since $\kappa \subset \mathcal{B}$ and the hereditariness of $\mathcal{B}$ implies $\pi \subset \mathcal{B}$, by $B \in \mathcal{K}$ we get that $B \in \mathcal{K} \cap \mathcal{B} = \mathcal{J}$, contradicting the primitivity of B. Thus $\mathcal{K} \subseteq \Xi$. The left ideal $L = Wy$ of the Weyl algebra W is a simple domain without nonzero idempotents, as it is well known. So $L \in \mathcal{B} \cap \mathcal{SJ}$ and $L \notin u$. Hence $L \notin \mathcal{J} = \mathcal{K} \cap \mathcal{B}$ and $L \in \Xi$ follow, implying $L \in \Xi \setminus \mathcal{K}$ and $\mathcal{K} \subset \Xi$.

Since every simple ring with unity element is in Ξ, we have $\Xi \not\subseteq \mathcal{U}\{S\}$. □

Theorem 4.5. *If A is a nil ring and $A[x] \in \Xi$, then $A[x] \in \mathcal{J}$.*

Proof. Since A is a nil ring, by [1] we have $A[x] \in \mathcal{B} \subset \mathcal{U}m$.

We shall show that $A[x] \in \mathcal{U}\nu_\gamma$. Assume that $A[x] \notin \mathcal{U}\nu_\gamma$. Then $A[x]$ has a nonzero homomorphic image B in $\mathcal{SJ}$, and so B has a nonzero primitive homomorphic image C. By Proposition 4.3 we have $A[x] \in \kappa$ and also $C \in \kappa$. Hence $A[x] \notin \Xi$, a contradiction. Thus

$$A[x] \in \mathcal{U}m \cap \mathcal{U}\nu_{\mathcal{J}} \subseteq \mathcal{U}\mu_{\mathcal{J}} \cap \mathcal{U}\nu_{\mathcal{J}} = \mathcal{J}$$

in view of Proposition 4.2. □

To attempt the finding of an explicit counterexample, the following may be helpful.

Corollary 4.6. *The following assertiong are equivalent:*

i) *Köthe's Problem has a positive solution;*
ii) *$A[x] \in \Xi$ for every nil ring A;*
iii) *$\ell(A[x]) = \Xi(A[x])$ for every nil ring A.*

Proof. i)$\Longleftrightarrow$ii) If Köthe's Problem has a positive solution, then we have $A[x] \in \mathcal{J} \subset \Xi$ for every nil ring A.

Suppose that $A[x] \in \Xi$ for every nil ring. Then by Theorem 4.5 we have $A[x] \in \mathcal{J}$.

ii)$\Longleftrightarrow$iii) This is obvious by Theorems 2.4 and 4.5. □

References

[1] Beidar K.I., Fong Y., Puczyłowski E.R. *A polynomial ring over a nil ring cannot be homomorphically mapped onto a ring with nonzero idempotents.* J. Algebra, 2001, **238**, p. 389–399.

[2] Beidar K.I., Puczyłowski E.R., Wiegandt R. *Radicals and polynomial rings.* J. Austral. Math. Soc., 2002, **71**, p. 1–7.

[3] Drazin M.P., Roberts M.L. *Polynomial, multiplicative and special radicals.* Comm. in Algebra, 2000, **28**, p. 3073–3093.

[4] Gardner B.J., Wiegandt R. *Radical theory of rings.* Marcel Dekker, 2004.

[5] Krempa J. *Logical connections among some open problems in non-commutative rings.* Fund. Math., 1972, **76**, p. 121–130.

[6] McConnell N.R., Stokes T. *Rings having simple adjoint semigroup.* Contemporary Math., 2001, **273**, p. 203–208.

[7] Sakhajev I.I. *Negative solution hypothesis of Koethe and some problems connected with it.* Abstract of Talks. Algebra Conference Venezia, 2002, p. 37–38.

[8] Sands A.D. *Radicals and Morita contexts.* J. Algebra, 1973, **24**, p. 335–345.

[9] Tumurbat S. *The radicalness of polynomial rings over nil rings.* Math. Pannonica, 2002, **13**, p. 191–199.

Received September 29, 2003

S. Tumurbat
Department of Algebra
University of Mongolia
P.O. Box 75, Ulaan Baatar 20
Mongolia
E-mail: *stumurbat@hotmail.com*

R. Wiegandt
A. Rényi Institute of Mathematics
Hungarian Academy of Sciences
P.O. Box 127, H-1364 Budapest
Hungary
E-mail: *wiegandt@renyi.hu*