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Radicals of rings with involution

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Abstract. The aim of the present paper is to give a survey of the most important features of radicals in associative rings with involution including some new remarks and the most recent results on primitivity.

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1 Basic definitions and important examples

Throughout the paper a *ring with involution* R will be an associative ring endowed with a supplementary operation $*: x \to x^*$ called involution and satisfying the rules

$$\begin{array}{rcl} (x^*)^* &=& x \\ (x+y)^* &=& x^* + y^* \\ (xy)^* &=& y^* x^* \,. \end{array}$$

An element x of R is called *symmetric* if $x^* = x$ and skew if $x^* = -x$; the sets of these elements will be denoted by S resp. K. The element $x + x^*$ is called the *trace* of x and $x - x^*$ the *skew-trace* of x. For every ring R, R^{op} will denote the ring obtained by interchanging the order of the elements in the multiplication.

The most important examples of involutions are:
- the trivial involution $x^* = x$ on commutative rings,
- the conjugate $(x + iy)^* = x - iy$ on the complex numbers,
- the additive inverse $x^* = -x$ on commutative rings,
- the exchange involution $(x, y)^* = (y, x)$ on $R \oplus R^{op}$,
- transposition $A^* = A^T$ on rings M_n of $n \times n$ matrices and
- the symplectic involution $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} D^T & -C^T \\ -B^T & A^T \end{pmatrix}$ on rings M_{2n} of $2n \times 2n$-matrices.

2 Some older results

The first results on rings with involution linked with radicals appear in the sixties and seventies of the 20^{th} century. They go back to J.M. Osborn, C. Lanski and S. Montgomery; the following generalized versions can be found in Herstein's book [6].

Theorem 2.1 ([6], 2.1.8). *A semiprime ring R with involution in which every nonzero trace $x + x^*$ is invertible is of one of the following four types:*

1. R is a commutative ring of characteristic 2 with no nonzero nilpotent elements endowed with the trivial involution;

2. R is a division ring with no restriction on the involution;

3. $R \simeq D \oplus D^{op}$ for some division ring D endowed with the exchange involution;

4. $R \simeq M_2$ over a field with the symplectic involution.

Further results of the same type are:

Theorem 2.2 ([6], 2.1.7). *A semiprime ring R with involution in which every nonzero symmetric element is invertible is of one of the types 2,3 and 4 of the preceding theorem.*

Theorem 2.3 ([6], 2.3.1). *A semiprime noncommutative ring R with involution in which every nonzero skew-trace $x - x^*$ is invertible is of one of the types 2,3 and 4 of theorem 2.1.*

Theorem 2.4 ([6], 2.3.2). *If R is a noncommutative ring with involution which is not nil-semisimple and in which every nonzero skew-trace $x - x^*$ is invertible, then its nil-radical $N(R)$ satisfies*

1. $R/N(R)$ is commutative and

2. $N(R)^2 = 0$.

Theorem 2.5 ([6], 2.3.4). *For a ring R with involution in which every trace $x+x^*$ is nilpotent or invertible (or in which every skew-trace $x-x^*$ is nilpotent or invertible) the factor ring $R/J(R)$ by its Jacobson-radical $J(R)$ is of one of the following four types:*

1. a commutative ring with trivial involution;

2. a division ring with no restriction on the involution;

3. $R/J(R) \simeq D \oplus D^{op}$ with the exchange involution for some division ring D;

4. $R/J(R) \simeq M_2$ over a field endowed with the symplectic involution.

Notice that the fact to have the symplectic involution in the last case (although it is not mentioned in [6]) follows from theorem 1 resp. 3 since in this case $R/J(R)$ is simple and Jacobson-semisimple, hence semiprime.

A careful look to all these results reveals that they are dealing with concepts defined in the variety of associative rings without involution and make use of the supplementary operation of involution just in order to get a more precise description of some objects.

3 Radical theory in the variety of associative rings with involution

The appropriate terms to build up a radical theory in the class of all associative rings with involution are those of homomorphisms compatible as well with the involution and the corresponding ideals.

We call these ring-homomorphisms (fulfilling as well $f(x)^* = f(x^*)$) $*$-*homomorphisms* and the corresponding kernels $*$-*ideals*; the latter are exactly those ideals I of a ring with involution R which satisfy $I^* = I$. R is called $*$-simple if it contains no nontrivial $*$-ideals and $*$-prime resp. $*$-semiprime if $I \cdot J = 0$ for $*$-ideals I and J of R implies $I = 0$ or $J = 0$ (resp.$I^2 = 0$ implies $I = 0$).

The difference to the classical concepts without involution is exhibited by the following

Proposition 3.1 (see for ex. [18]).

1. A ring R with involution is $*$-simple if and only if R is simple or $R \simeq S \oplus S^{op}$ for some simple ring S, the involution being the exchange involution.

2. R is $*$-prime if and only if R contains a prime ideal P satisfying $P \cap P^* = 0$, i.e. if and only if R is prime or a subdirect product of two prime rings.

Remark 3.2. *However, a ring R with involution is $*$-semiprime if and only if it is semiprime.*

Radical theory in the variety of associative rings with involution was introduced in 1977 by Salavova [19]. She pointed out that the general radical theory introduced by Kurosh [7] and Amitsur [2] for Ω-groups resp. by Ryabuhin [17] for certain categories applies to the variety of rings with involution. The obtained radicals in this class will be called $*$-*radicals*. Consequently the following assertions hold:

Theorem 3.3.

1. (see for ex. [21]) *A mapping ρ assigning to every ring R with involution a $*$-ideal ρR is a $*$-radical if and only if the following conditions hold:*

($\rho 1$) $f(\rho R) \subseteq \rho(fR)$ *for every $*$-homomorphisms f defined on R*

($\rho 2$) $\rho(R/\rho R) = 0$

($\rho 3$) *ρ is idempotent:* $\rho(\rho R) = \rho R$

($\rho 4$) *ρ is complete:* $I \lhd^* R, \rho I = I \Rightarrow I \subseteq \rho R$.

2. (see [15]) *A class $\mathbf{R}$ of rings with involution is the radical class of a $*$-radical if and only if the following assertions hold:*

($\mathbf{R}1$) $\mathbf{R}$ *is closed under taking $*$-homomorphic images;*

($\mathbf{R}2$) $\mathbf{R}$ *is closed under taking sums of $*$-ideals within rings with involution;*

($\mathbf{R}3$) $\mathbf{R}$ *is $*$-extension closed, i.e. $I \lhd^* R$ with $I \in \mathbf{R}$ and $R/I \in \mathbf{R}$ implies* $R \in \mathbf{R}$.

3. (see [15]) *A class $\mathbf{S}$ of rings with involution is the semisimple class of a $*$-radical if and only if the following assertions hold:*

($\mathbf{S}1$) $\mathbf{S}$ *is closed under $*$-subdirect products, i.e. $I_\lambda \lhd^* R$ with $\bigcap_{\lambda\in\Lambda} I_\lambda = 0$ and $R/I_\lambda \in \mathbf{S}$ for all $\lambda \in \Lambda$ implies $R \in \mathbf{S}$;*

($\mathbf{S}2$) $\mathbf{S}$ *is $*$-extension closed;*

($\mathbf{S}3$) $\mathbf{S}$ *is $*$-regular, i.e. $0 \neq I \lhd^* R \in \mathbf{S}$ implies that I has a nontrivial $*$-homomorphic image in* $\mathbf{S}$;

($\mathbf{S}4$) $(R\mathbf{S})\mathbf{S} \lhd^* R$, *where $R\mathbf{S}$ denotes the intersection of all $*$-ideals I of R with* $R/I \in \mathbf{S}$.

Remark 3.4. *Notice that by Salavova's example 1.9 we know that the semisimple classes of $*$-radicals are not necessarily $*$-hereditary (i.e. $I \lhd R \in \mathbf{S}$ does not imply $I \in \mathbf{S}$). Hence, condition* ($\mathbf{S}3$) *cannot be replaced by heredity as in the case of associative rings without involution and* ($\mathbf{S}4$) *can not be omitted as far as is known.*

By the above remark it is clear that $*$-radicals do not always have the ADS-property, i.e. the $*$-radical of a $*$-ideal I of R is not necessarily an ideal in R (it is obviously closed under the involution). This fact lead to a series of papers by Loi and Wiegandt containing the following main results:

Theorem 3.5 ([13]). *For a $*$-radical ρ on the variety of all algebras with involution over a commutative ring R with identity the following assertions are equivalent:*

(1) ρ has the ADS-property;

(2) if an algebra A with involution belongs to the radical class $\mathbf{R}_\rho$ of ρ and satisfies $A^2 = 0$ then A belongs to $\mathbf{R}_\rho$ when endowed with any other involution;

(3) A with $*$ belongs to $\mathbf{R}_\rho$ if and only if A with involution $x \to -x^*$ belongs to $\mathbf{R}_\rho$ whenever $A^2 = 0$;

(4) A with the trivial involution belongs to $\mathbf{R}_\rho$ if and only if A with the additive inverse involution $x \to -x$ belongs to $\mathbf{R}_\rho$ whenever $A^2 = 0$.

Remark 3.6. *An example constructed in [13] shows that a $*$-radical with a $*$-hereditary semisimple class does not necessarily have the ADS-property.*

Theorem 3.7 ([13]). *All $*$-radicals on the variety of involution algebras over a field K have the ADS-property if and only if char $K = 2$.*

A result of the same kind as above has been proved for algebras with involution over commutative rings R with involution and identity, the difference to the earlier case being the rule

$$(ra)^* = r^*a^* \text{ instead of } (ra)^* = ra^*$$

for all $r \in R$ and $a \in A$.

Theorem 3.8 ([9]). *On the class of all algebras with involution on a commutative ring with involution and identity the assertions (1) and (3) of theorem 3.5 are equivalent.*

Let us recall that a $*$-radical ρ of rings or algebras with involution is called *hypernilpotent* if every nilpotent ring, resp. algebra is in the radical class $\mathbf{R}_\rho$ and *hypoidempotent* if the radical class $\mathbf{R}_\rho$ consists of idempotent rings resp. algebras only.

Loi proved the following

Theorem 3.9 ([10]). *Every $*$-radical of algebras with involution over a field with nontrivial involution is either hypernilpotent or hypoidempotent.*

Using this result, he also obtained

Theorem 3.10 ([10]). *In the variety of all algebras with involution over a field with nontrivial involution every $*$-radical has the ADS-property.*

A description of classes which are both radical and semisimple with respect to suitable radicals can be found in a paper by Loi dating back to 1989:

Theorem 3.11 ([11]). *For radical-semisimple classes of involution algebras over a field K with involution the following assertions hold*

1. *If K is infinite, there are no nontrivial radical-semisimple classes*;
2. *If K is finite, every nontrivial semisimple (and hence every nontrivial radical-semisimple class) consists of all subdirect sums of algebras belonging to some strongly hereditary finite set of simple involution algebras.*

4 The connection between ring radicals and $*$-radicals

The next question arising is whether a radical of associative rings is already a $*$-radical for associative rings with involution. A complete answer was given in 1992 in a paper by Lee and Wiegandt:

Theorem 4.1 ([8]). *For a radical ρ of associative rings the following assertions are equivalent:*

1. *ρ is a $*$-radical, i.e. $\rho R \triangleleft^* R$ for every ring R with involution*;
2. *$R \in \mathbf{S}_\rho$ implies $R^{op} \in \mathbf{S}_\rho$*;
3. *$R \in \mathbf{R}_\rho$ implies $R^{op} \in \mathbf{R}_\rho$.*

This theorem infers that the ring radicals which are $*$-radicals are exactly the symmetric ones. (Notice that the basic definition needs not to be symmetric as can be seen from the Jacobson radical defined via primitivity). A list of the most important among them is given in

Corollary 4.2 ([8]). *The following ring radicals are $*$-it radicals: the Koethe (nil) radical, the generalized nil radical, the Baer (prime) radical, the Behrens radical, the Brown-McCoy radical, the Jacobson radical, the Levitzki radical, the von Neumann-regular radical, the strongly regular radical, the idempotent radical.*

Remark 4.3. *There are ring radicals which are not $*$-radicals as can be seen from the examples of the right strongly prime resp. the right superprime radical (see [16, 20]).*

In two papers from 1996 resp 1998, Booth and Groenewald looked to the question of constructing $*$-radicals from ring radicals. They introduced a mapping λ assigning to every ring radical ρ a $*$-radical λ_ρ taking for $(\lambda\rho)(R)$ the sum of all $*$-ideals I of R belonging to $\mathbf{R}_\rho$.

Theorem 4.4 ([5]). *The following assertions hold:*

1. *every ring radical ρ induces a $*$-radical λ_ρ*;
2. *λ maps the symmetric ring radicals bijectively onto the invariant $*$-radicals*, i.e. *those $*$-radicals for which $(\lambda\rho)(R)$ is the same for all involutions on R*;
3. *$(\lambda\rho)(R) \subseteq \rho R \cap \rho(R^{op})$ with equality whenever $\mathbf{R}_\rho$ is hereditary.*

Notice that in view of theorem 4.1, λ restricted to the symmetric ring radicals is in fact the identity mapping and yields therefore $*$-radicals with $*$-hereditary semisimple classes.

The dual to the above construction using the semisimple homomorphic images of R instead of the radical ideals has not been considered so far; thus for every ring radical ρ and every ring R with involution let us define

$$(\sigma\rho)(R) = \bigcap(K \lhd^* R | R/K \in \mathbf{S}_\rho).$$

Theorem 4.5. *The following assertions hold:*

1. *every ring radical ρ induces a $*$-radical $\sigma\rho$ with a $*$-hereditary semisimple class;*
2. *σ restricted to the symmetric ring radicals is the identity mapping;*
3. *$\rho R + \rho(R^{op}) \subseteq (\sigma\rho)(R)$ with equality whenever $\mathbf{S}_\rho$ is homomorphically closed.*
4. *$(\lambda\rho)R \subseteq \rho R \subseteq (\sigma\rho)(R)$ with equality if and only if ρ is symmetric.*

Proof.

1. We show that the class $\mathbf{S}_{\sigma\rho}$ is closed under taking subdirect products, extensions and $*$-ideals, the crucial point being the obvious inclusion $\rho R \subseteq (\sigma\rho)(R)$ with equality for $R \in \mathbf{S}_\rho$ since 0 is always a $*$-ideal.

If R is a subdirect product of involution rings $R_i \in \mathbf{S}_{\sigma\rho}(i \in I)$, then R, considered as a ring, is a subdirect product of the rings $R_i \in \mathbf{S}_\rho$ and thus belongs to $\mathbf{S}_\rho$; hence $\mathbf{S}_{\sigma\rho}$ is subdirectly closed.

If R and R/I belong to $\mathbf{S}_{\sigma\rho}$ for some $*$-ideal I of R, then R and R/I are rings belonging to $\mathbf{S}_\rho$; thus R belongs to $\mathbf{S}$ inferring $R \in \mathbf{S}_{\sigma\rho}$.

Heredity of $\mathbf{S}_{\sigma\rho}$ is obtained by a similar argument.

2. This assertion follows from the fact that the symmetric ring radicals are $*$-radicals.

3. $(\sigma\rho)(R)$ being a $*$-ideal, it contains both ρR and $(\rho R)^*$. The isomorphism $(\rho R)^* \simeq (\rho R)^{op} = \rho(R^{op})$ yields $\rho R + \rho(R^{op}) \subseteq (\sigma\rho)(R)$.

If $\mathbf{S}_\rho$ is homomorphically closed, then $R/\rho R + \rho(R^{op})$ belongs to $\mathbf{S}_\rho$, hence to $\mathbf{S}_{\sigma\rho}$ implying $(\sigma\rho)(R) \subseteq \rho R + \rho(R^{op})$.

4. Is a direct consequence of 2 and 3 in theorems 4.3 and 4.5.

Special ring radicals are defined by the upper radicals $\mathcal{U}M = \{R | R$ has no nonzero homomorphic image in $M\}$ of special classes M of rings, where a class M is called *special* if

1. M consists of prime rings,
2. M is hereditary,
3. M is closed under essential extensions, i.e. if an essential ideal I of R belongs to M, then R belongs to M.

Already Salavova introduced the concept of $*$-*special classes* of rings with involution in her paper [19]. Her definition is equivalent to the $*$-analogue of the above one obtained by writing $*$-prime, $*$-hereditary and essential $*$-ideal in 1,2 and 3.

In the definition of $*$-special radicals, we have to use $\mathcal{U}^* M = \{R | R$ has no nonzero $*$-homomorphic image in $M\}$.

In 1996, resp.1998 Booth and Groenewald showed how special classes induce $*$-special classes.

Theorem 4.6 ([4], resp. [5]). *Every special class M of rings induces a $*$-special class M of rings R with involution by*

$$M^* = \{R | \exists P \triangleleft R \quad with \quad P \cap P^* = 0 \quad and \quad R/P \in M\}.$$

Moreover, if ρ is a special radical of rings, then

$$\mathcal{U}^*(\mathbf{R}\rho) = \mathbf{R}_{\lambda\rho}$$

where λ is the mapping from theorem 4.4.

5 The role of $*$-biideals

One-sided ideals are the basis for the construction of many radicals of rings. However, in rings with involution $*$, they are never closed under $*$ unless they are two-sided, i.e. $*$-ideals. So, what kind of substructure could replace the one-sided ideals in presence of an involution?

Taking just the $*$-ideals, the situation would become similar to that of commutative rings and we would loose a lot of information. A much better solution is inherited by considering the fact that in the classical case the annihilators of elements of R-modules are one-sided ideals of R. For (one-sided) modules over rings R with involution, it is usual to give the following definition:

$$Ann^*_R m = \{r \in R | rm = 0 = r^* m\} \quad \text{and}$$
$$Ann^*_R M = \{r \in R | rM = 0 = r^* M\}.$$

It is easy to see that for $Ann^*_A M$ instead of an ideal in the classical case we now obtain a $*$-ideal.

$Ann^*_R m$ can be seen either as a $*$-*quasiideal*, i.e. a subgroup Q of $(R, +)$ satisfying $Q^* = Q$ and $QR \cap RQ \subseteq Q$ or as a $*$-*biideal*, i.e. a subgroup B of $(R, +)$ which satisfies $B^* = B$ and $BRB \subseteq B$. Since quasiideals are not necessarily subrings, the suitable structure to replace one-sided ideals in the theory of $*$-radicals seems to be that of $*$-biideals.

The first result underlining this conjecture has been given by Loi in 1990:
Theorem 5.1 ([12]). *A semiprime ring R with involution has d.c.c on principal right ideals (when considered just as a ring) if and only if R has d.c.c. on principal $*$-biideals (i.e. $*$-biideals generated by a single element).*

This theorem was generalised by Aburawash in 1991:

Theorem 5.2 ([1]). *A semiprime ring R with involution has d.c.c. on right ideals (when considered as a ring) if and only if it has d.c.c. on $*$-biideals. Moreover, such a ring always has a.c.c.on $*$-biideals as well.*

In 1993, Beidar and Wiegandt proved

Theorem 5.3 ([3]). *A ring R with involution has d.c.c on $*$-biideals if and only if both R and its Jacobson radical $J(R)$ are right- and left-artinian, i.e. have d.c.c. on right and left ideals.*

Remark 5.4. *In the same paper an example of a ring with involution is given which, considered just as a ring, has both a.c.c. on right and left ideals, but fails to have a.c.c. on $*$-biideals.*

Recently, $*$-primitive rings with involution have been studied. A ring R with involution is called *$*$-primitive* if there is an irreducible R-left-module M satisfying $Ann^*_R M = 0$.

It is well known (see for ex. [18]) that a ring with involution is $*$-primitive if and only if considered without involution it is either a left-primitive ring or the subdirect sum $R/P \oplus_{sub} R/P^*$ of a left and a right primitive ring.

A subdirect sum giving rather poor information, it seemed worth to look for a new description of $*$-primitivity.

Let us recall that a ring R (without involution) is called *primitive* if there is a faithful irreducible R-module, i.e. if R contains a maximal left ideal L such that $Ann_R R/L = 0$. $Ann_R R/L$ being the largest ideal of R contained in L, this means that R contains a maximal left ideal L which does not contain any nonzero ideal of R.

Now, the $*$-analogue is given by:

Theorem 5.5 ([14]). *A ring R with involution is $*$-primitive if and only if it contains a maximal $*$-biideal which does not contain any nonzero $*$-ideal of R.*

Furthermore, in the same paper, the involution analogue of the well known statement that a prime ring with a minimal left ideal is primitive has been proved.

Theorem 5.6 ([14]). *A $*$-prime ring with involution with a minimal $*$-biideal is $*$-primitive.*

Thus, the suitable structure to use in rings with involution instead of one-sided ideals are the $*$-biideals.

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