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Exponent matrices and their quivers

V.V. Kirichenko, A.V. Zelensky, V.N. Zhuravlev

Abstract. We consider exponent matrices and investigate their connections with tiled orders and quivers, finite partially ordered sets and doubly stochastic matrices.

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1 Introduction

Exponent matrices appeared in the study of tiled orders over discrete valuation rings. Many properties of such orders are formulated using this notion. We think that such matrices are of interest in them own right, in particular, it is convenient to write finite partially ordered sets (posets) and finite metric spaces as special exponent matrices.

Note that when we defined a quiver $Q(\mathcal{E})$ of a reduced exponent matrix $\mathcal{E}$, $\mathcal{E}$ corresponds to a reduced tiled order Λ, a matrix $\mathcal{E}^{(1)}$ corresponds to a Jacobson radical R of Λ, and $\mathcal{E}^{(2)}$ corresponds to R^2. Then the adjacency matrix $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ defines a structure of the Λ-bimodule $V = R/R^2$.

Note that investigations on tiled orders over discrete valuation rings and finite posets are discussed in [10]. The bibliography about tiled orders see in [2] and [3].

2 Quivers

We recall basic facts about quivers and related topics. Following P. Gabriel a finite directed graph Q is called *a quiver*.

Definition 2.1. *A quiver Q without multiple arrows and multiple loops is called a simply laced quiver.*

Denote by $VQ = \{1, \dots, s\}$ the set of all vertices of Q and by AQ the set of its all arrows. We shall write $Q = \{AQ, VQ\}$. Denote by $1, \dots, s$ the vertices of a quiver Q and assume that we have q_{ij} arrows beginning at the vertex i and ending at the vertex j. The matrix

$$[Q] = \begin{pmatrix} q_{11} & q_{12} & \dots & q_{1s} \\ q_{21} & q_{22} & \dots & q_{2s} \\ \dots & \dots & \dots & \dots \\ q_{s1} & q_{s2} & \dots & q_{ss} \end{pmatrix}$$

is called *the adjacency matrix* of Q.

Obviously, a quiver Q is simply laced if and only if $[Q]$ is a $(0,1)$-matrix.

Let Q be a quiver. Usually we will denote the vertices of Q by the numbers $1, 2, \ldots, s$. If an arrow σ connects a vertex i with a vertex j then i is called its *start vertex* and j its *end vertex*. This will be denoted as $\sigma : i \to j$. A loop at the vertex j is an arrow such that the start vertex j coincides with the end vertex j.

A *path of the quiver* Q from a vertex i to a vertex j is an ordered set of k arrows $\{\sigma_1, \sigma_2, ..., \sigma_k\}$ such that the start vertex of each arrow σ_m coincides with the end vertex of the previous one σ_{m-1} for $1 < m \leq k$, and moreover, the vertex i is the start vertex of σ_1, while the vertex j is the end vertex of σ_k. The number k of these arrows is called the *length of the path*.

The start vertex i of the arrow σ_1 is called the *start of the path* and the end vertex j of the arrow σ_k is called the *end of the path*. We shall say that the path connects the vertex i with the vertex j and it is denoted by $\sigma_1\sigma_2...\sigma_k : i \to j$.

Now we shall give a definition of a diagram $Q(P)$ of a finite poset P.

Definition 2.2. ([1], Ch.1, §3). *By "a covers b" in a poset P, it is meant that $a > x > b$ for no $x \in P$.*

Definition 2.3. ([4], p. 233, see also [6]). *Let $P = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a finite poset with an ordering relation $\leq$. The* diagram of P *is the quiver $Q(P)$ with the set of vertices $VQ(P) = \{1, \ldots, n\}$ and the set of arrows $AQ(P)$ such that in $AQ(P)$ there is an arrow $\sigma : i \to j$ if and only if α_j covers α_i.*

Definition 2.4. ([7], §8.4). *A quiver without oriented cycles is called an* acyclic quiver.

Definition 2.5. *An arrow $\sigma : i \to j$ of an acyclic quiver Q is called* extra *if there exists a path from i to j of length greater than 1.*

Theorem 2.6. ([6], [4], §7.7). *Let Q be an acyclic simply laced quiver without extra arrows. Then Q is the diagram of some finite poset P. Conversely, the diagram $Q(P)$ of a finite poset P is an acyclic simply laced quiver without extra arrows.*

3 Exponent matrices

Denote by $M_n(\mathbb{Z})$ the ring of all square $n \times n$-matrices over the ring of integers $\mathbb{Z}$. Let $\mathcal{E} \in M_n(\mathbb{Z})$.

Definition 3.1. *We call a matrix $\mathcal{E} = (\alpha_{ij})$ an* exponent matrix *if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for $i, j, k = 1, \ldots, n$ and $\alpha_{ii} = 1, \ldots, n$ for $i = 1, \ldots, n$. These relations are called* ring inequalities. *An exponent matrix $\mathcal{E}$ is called* reduced *if $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \ldots, n$.*

Let $\mathcal{E} = (\alpha_{ij})$ be a reduced exponent matrix. Set $\mathcal{E}^{(1)} = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ and $\beta_{ii} = 1$ for $i = 1, \ldots, n$, and $\mathcal{E}^{(2)} = (\gamma_{ij})$, where $\gamma_{ij} = \min\limits_{1 \leq k \leq n} (\beta_{ik} + \beta_{kj})$. Obviously, $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is a $(0,1)$-matrix.

Definition 3.2. *The quiver $Q(\mathcal{E})$ shall be called the quiver of the reduced exponent matrix $\mathcal{E}$.*

Definition 3.3. *A strongly connected simply laced quiver shall be called* admissible *if it is a quiver of a reduced exponent matrix.*

Definition 3.4. *A reduced exponent matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ shall be called* Gorenstein *if there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \ldots, n$.*

The permutation σ is denoted by $\sigma(\mathcal{E})$. Notice that $\sigma(\mathcal{E})$ for a reduced Gorenstein exponent matrix $\mathcal{E}$ has no cycles of the length 1.

Definition 3.5. *We shall call two exponent matrices $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$* equivalent *if they can be obtained from each other by transformations of the following two types:*

(1) subtracting an integer from the i-th row with simultaneous adding it to the i-th column;

(2) simultaneous interchanging of two rows and the equally numbered columns.

Proposition 3.6. [3]. *Suppose that $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ are exponent matrices and Θ is obtained from $\mathcal{E}$ by a transformation of type (1). Then $[Q(\mathcal{E})] = [Q(\Theta)]$. If $\mathcal{E}$ is a reduced Gorenstein exponent matrix with permutation $\sigma(\mathcal{E})$, then Θ is also reduced Gorenstein with $\sigma(\Theta) = \sigma(\mathcal{E})$.*

Proposition 3.7. [3]. *Under transformations of the second type the adjacency matrix $[\tilde{Q}]$ of $Q(\Theta)$ changes according to the formula: $[\tilde{Q}] = P_\tau^T[Q]P_\tau$, where $[Q] = [Q(\mathcal{E})]$. If $\mathcal{E}$ is Gorenstein then Θ is also Gorenstein and for the new permutation π we have: $\pi = \tau^{-1}\sigma\tau$, i.e., $\sigma(\Theta) = \tau^{-1}\sigma(\mathcal{E})\tau$.*

Definition 3.8. *The* index *(in $\mathcal{E}$) of a reduced exponent matrix $\mathcal{E}$ is the maximal real eigenvalue of the adjacency matrix $[Q(\mathcal{E})]$ of $Q(\mathcal{E})$.*

It follows from Proposition 3.6 and Proposition 3.7 that indices of equivalent reduced exponent matrices coincide.

Theorem A. *The matrix $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is the adjacency matrix of the strongly connected simply laced quiver $Q = Q(\mathcal{E})$.*

Proof. $[Q]$ is a $(0,1)$-matrix, then it is the adjacency matrix of a simply laced quiver.

We shall show that $[Q]$ is a strongly connected quiver. Suppose the contrary. It means that there is no path from the vertex i to the vertex j in Q. Denote by $VQ(i) = V_1$ the set of all vertices k of Q such that there exists a path beginning at the vertex i and ending at the vertex k. It is obviously that $V_2 = VQ \setminus VQ(i) \neq 0$ ($j \in V(Q) \setminus V(Q)(i)$). Consequently, $VQ = V_1 \cup V_2$ and $V_1 \cap V_2 = 0$. It is clear that there are no arrows from V_1 to V_2. One can assume that $V_1 = \{1, \ldots, m\}$ and $V_2 = \{m+1, \ldots, s\}$. It is obvious, that a simultaneous permutation of rows and columns will take place in the exponent matrix $\mathcal{E}$. Moreover, under transformations of the first type, we can make the elements at the first row of $\mathcal{E}$ equal zero, i.e., $\alpha_{1p} = 0$ for $p = 1, \ldots, s$. So, $\alpha_{pq} \geq 0$ for $p, q = 1, \ldots, s$ and

$$[Q] = \left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right),$$

$$\mathcal{E} = \left(\begin{array}{c|c} \mathcal{E}_1 & * \\ \hline * & \mathcal{E}_2 \end{array}\right),$$

where $\mathcal{E}_1 \in M_m(\mathbb{Z})$, $\mathcal{E}_2 \in M_{s-m}(\mathbb{Z})$. With the exponent matrix $\mathcal{E}_2$ we connect a poset $P_{\mathcal{E}_2} = \{m+1, \ldots, s\}$ with an ordering relation $i \leq j$ if and only if $\alpha_{ij} = 0$. One can consider that $m+1 \in P_{\mathcal{E}_2}$ is the minimal element. Then $\alpha_{im+1} > 0$ for $i > m+1$. Since, $q_{1m+1} = 0$, then there exists k $(2 \leq k \leq m)$ such that $\alpha_{1m+1} = \alpha_{1k} + \alpha_{km+1}$. Simultaneously interchanging the 2-nd and k-th columns and the 2-nd and k-th rows of $\mathcal{E}$, we obtain that $\alpha_{2m+1} = 0$. Since $q_{2m+1} = 0$, again obtain $\alpha_{2m+1} = 0 = \alpha_{2k} + \alpha_{2m+1}$ for $3 \leq k \leq m$, i.e., one can consider that $\alpha_{23} = 0$ and $\alpha_{3m+1} = 0$. The elements of the matrix $\mathcal{E}^{(1)}$ $\beta_{31} = \alpha_{31}$, $\beta_{32} = \alpha_{32}$, $\beta_{33} = 1$ are nonzero. Again, $q_{3m+1} = 0$ and $\alpha_{3m+1} = 0 = \alpha_{3k} + \alpha_{km+1}$ for $4 \leq k \leq m$. Hence, $\alpha_{4m+1} = 0$. Continuing this process we have that $\alpha_{12} = \alpha_{23} = \ldots = \alpha_{m-1m} = 0$ and $\alpha_{im+1} = 0$ for $i = 1, \ldots, m$, consequently a matrix $\mathcal{E}_1$ is down triangular, and all elements $\beta_{m1}, \ldots, \beta_{mm}$ are natural integers. So, $q_{mm+1} = \min(\beta_{mk} + \beta_{km+1}) - \alpha_{mm+1} = 1 - 0 = 0$. We obtained a contradiction. Theorem is proved. □

4 Gorenstein exponent matrices and entropic quasigroups

In general case a Latin square [5] of order n is a square with rows and columns each of which is a permutation of a set $S = \{s_1, \ldots, s_n\}$. Every Latin square is a Cayley table of a finite quasigroup. In particular, the Cayley table of a finite group is the Latin square. As a set S we will consider $S = \{0, 1, \ldots, n-1\}$.

Example 1. The Cayley table of the Klein four-group $(2) \times (2)$ can be written in such form:

$$K = K(4) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$

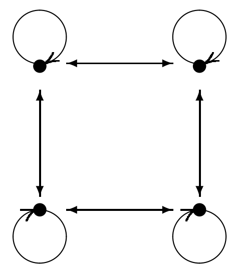
Then $K(4)$ is a reduced Gorenstein exponent matrix with permutation $\sigma = \sigma(K(4)) = (14)(23)$. Obviously,

$$K^{(2)} = \begin{bmatrix} 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 3 & 3 & 2 & 2 \\ 3 & 3 & 2 & 2 \end{bmatrix}$$

and

$$[Q(K)] = K^{(2)} - K^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = 3 \cdot P_1,$$

where P_1 is a doubly stochastic matrix, and $Q(K)$ is



Obviously, $in\, K = 3$.

Definition 4.1. *A real non-negative $s \times s$-matrix $P = (p_{ij})$ is doubly stochastic if $\sum_{j=1}^{s} p_{ij} = 1$ and $\sum_{i=1}^{s} p_{ij} = 1$ for any $i, j = 1, \ldots, s$.*

Definition 4.2. (see [8], p. 140). *A quasigroup Q which satisfies the identity $(xu)(vy) = (xv)(uy)$ for $x, y, u, v \in Q$ is called entropic.*

Example 2. ([8], p. 141, V. 2.2.1. Example). Let $Q(5) = \{0, 1, 2, 3, 4\}$ be the quasigroup with the following Cayley table

0	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

It is clear, that $Q(5)$ is an entropic quasigroup. The Cayley table

$$\mathcal{E}(5) = \begin{bmatrix} 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

of $Q(5)$ is a reduced Gorenstein exponent matrix with $\sigma(\mathcal{E}(5)) = (12345)$.

Obviously,

$$[Q(\mathcal{E}(5))] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = 2P_2,$$

where P_2 is a doubly stochastic matrix, and $in\,\mathcal{E}(5) = 2$.

Definition 4.3. *A reduced Gorenstein exponent matrix $\mathcal{E}$ is called cyclic if $\sigma(\mathcal{E})$ is a cycle.*

Remark. *Note, that a reduced tiled order Λ is Gorenstein if and only if its reduced exponent matrix $\mathcal{E}(\Lambda)$ is Gorenstein.*

Hence, in view of the Theorem 3.4 [9] we have such theorem.

Theorem B. *Let $\mathcal{E}$ be a cyclic reduced Gorenstein exponent matrix. Then $[Q(\mathcal{E})] = \lambda P$, where λ is a positive integer and P is a doubly stochastic matrix.*

For the Cayley table

$$\mathcal{E}(n) = \begin{bmatrix} 0 & n-1 & n-2 & \dots & 2 & 1 \\ 1 & 0 & n-1 & \dots & 3 & 2 \\ 2 & 1 & 0 & \dots & 4 & 3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ n-2 & n-3 & n-4 & \dots & 0 & n-1 \\ n-1 & n-2 & n-3 & \dots & 1 & 0 \end{bmatrix}$$

of the entropic quasigroup $Q(n)$, we have $[Q(\mathcal{E}(n))] = E_n + J_n^-(0) + e_{1n}$, where $J_n^-(0) = e_{21} + \ldots + e_{nn-1}$ is the lower nilpotent Jordan block.

The next definition is given in ([9], Section IV).

Definition 4.4. *A finite quasigroup Q defined on the set $S = \{0, 1, \ldots, n-1\}$ is called Gorenstein if its Cayley table $C(Q) = (\alpha_{ij})$ has a zero main diagonal and there exists a permutation $\sigma : i \to \sigma(i)$ for $i = 1, \ldots, n$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i = 1, \ldots, n$.*

If σ is a cycle then G is a cyclic Gorenstein quasigroup.

Proposition 4.5. *The quasigroup $Q(n)$ is Gorenstein with permutation $\sigma = (12 \ldots n)$, i.e. $Q(n)$ is a cyclic Gorenstein quasigroup.*

Proof. Obvious.

Theorem 4.6. *For any permutation $\sigma \in S_n$ without fixed elements there exists a Gorenstein reduced exponent matrix $\mathcal{E}$ with permutation $\sigma(\mathcal{E}) = \sigma$.*

Proof. Suppose that σ has no cycles of length 1 and decomposes into a product of non-intersecting cycles $\sigma = \sigma_1 \cdots \sigma_k$, where σ_i has length m_i. Denote by t the least common multiple of the numbers $m_1 - 1, \ldots, m_k - 1$.

Consider the matrix

$$\mathcal{E}(m_1,\dots,m_s)=\begin{pmatrix} t_1\mathcal{E}(m_1) & tU_{m_1\times m_2} & tU_{m_1\times m_3} & \dots & tU_{m_1\times m_k}\\ 0 & t_2\mathcal{E}(m_2) & tU_{m_2\times m_3} & \dots & tU_{m_2\times m_k}\\ 0 & 0 & t_3\mathcal{E}(m_3) & \dots & tU_{m_3\times m_k}\\ \dots & \dots & \dots & \ddots & \dots\\ 0 & 0 & 0 & \dots & t_k\mathcal{E}(m_k)\end{pmatrix},$$

where $t_j=\frac{t}{m_j-1}$, $U_{m_i\times m_j}$ is an $m_i\times m_j$ - matrix whose entries equal 1; $\mathcal{E}(m)=(\varepsilon_{ij})$, $\varepsilon_{ij}=\begin{cases} i-j, & \text{if } i\geq j;\\ i-j+m, & \text{if } i<j.\end{cases}$

Let us remark that $\varepsilon_{ij}+\varepsilon_{j\sigma(i)}=\varepsilon_{i\sigma(i)}=m-1$ for all i,j.

Evidently, $\mathcal{E}(m_1,\dots,m_s)$ is the reduced Gorenstein exponent matrix with permutation $\pi(A)=(123\dots m_1)(m_1+1\dots m_1+m_2)\cdots(m_1+m_2+\dots+m_{k-1}+1\dots m_1+m_2+\dots+m_{k-1}+m_k)$.

Since the permutations σ and π have the same type, these permutations are conjugate, i.e., there exists a permutation τ such that $\sigma=\tau^{-1}\pi(A)\tau$.

Consequently, by Propositions 3.6 and 3.7, the matrix $P_\tau^T\mathcal{E}(m_1,\dots,m_s)P_\tau$ is the reduced Gorenstein exponent matrix with permutation $\sigma(\mathcal{E})=\sigma$. □

In conclusion of this section we formulate the following question.

Suppose that a Latin square $\mathcal{E}$ [5] defined on $S=\{0,1,\dots,n-1\}$ is an exponent matrix which is doubly symmetric, that is $\mathcal{E}$ is symmetric with respect to the main diagonal and is also symmetric with respect to the secondary diagonal. Suppose also that the first row of $\mathcal{E}$ is $\{0\,1\,2\,\dots\,n-1\}$.

Is it true that $\mathcal{E}$ is necessarily the Cayley table of an elementary abelian 2-group?

5 Reduced exponent $(0,1)$-matrices and finite partially ordered sets

With any finite partially ordered set (poset) P we relate a reduced exponent $(0,1)$-matrix $\mathcal{E}_P=(\lambda_{ij})$ by the following way: $\lambda_{ij}=0\Leftrightarrow i\leq j$, otherwise $\lambda_{ij}=1$.

It is easy to see that $\mathcal{E}_P$ is indeed a reduced exponent matrix.

Conversely, a reduced $(0,1)$-matrix $\mathcal{E}=(\lambda_{ij})$ defines the finite poset $P_{\mathcal{E}}$ by the rule: $i\leq j$ if and only if $\lambda_{ij}=0$, and $P_{\mathcal{E}_P}=P$.

Denote by P_{max} (resp. P_{min}) the set of the maximal (resp. minimal) elements of P and by $P_{max}\times P_{min}$ their Cartesian product.

From ([2], Theorem 6.12) we have

Theorem C. *The quiver $Q(\mathcal{E}_P)$ can be obtained from the diagram $Q(P)$ by adding the arrows σ_{ij} for all $(p_i,p_j)\in P_{max}\times P_{min}$.*

Definition 5.1. *We shall say that finite posets S and T are Q-equivalent if reduced exponent $(0,1)$-matrices $\mathcal{E}_S$ and $\mathcal{E}_T$ are equivalent.*

Definition 5.2. *An index in P of a finite poset P is the maximal real eigen-value of the adjacency matrix $[Q(\mathcal{E}_P)]$ of $Q(\mathcal{E}_P)$.*

Now we shall give the list of indexes of posets with at most four elements.

I. $(1) = \{\bullet\}$, $in\,(I,1) = 1$.

II. $(1) = \left\{\begin{array}{c}\bullet\\ |\\ \bullet\end{array}\right\}$, $in\,(II,1) = 1$; $(2) = \{\bullet\ \bullet\}$, $in(II,2) = 2$.

III. $(1) = \left\{\begin{array}{c}\bullet\\ |\\ \bullet\\ |\\ \bullet\end{array}\right\}$, $in\,(III,1) = 1$;

$(2) = \left\{\begin{array}{ccccc} & & \bullet & & \\ & / & & \backslash & \\ \bullet & & & & \bullet\end{array}\right\}$, $(3) = \left\{\begin{array}{ccccc}\bullet & & & & \bullet\\ & \backslash & & / & \\ & & \bullet & & \end{array}\right\}$, $in\,(III,2) = in(III,3) = \sqrt{2}$;

$(4) = \left\{\begin{array}{cc} & \bullet\\ & |\\ \bullet & \bullet\end{array}\right\}$, $in\,(III,4) = \frac{1+\sqrt{5}}{2}$; $(5) = \{\bullet\ \bullet\ \bullet\}$, $in(III,5) = 3$.

IV. $(1) = \left\{\begin{array}{c}\bullet\\ |\\ \bullet\\ |\\ \bullet\\ |\\ \bullet\end{array}\right\}$, $in\,(IV,1) = 1$; $(2) = \left\{\begin{array}{ccccc} & & \bullet & & \\ & / & & \backslash & \\ \bullet & & & & \bullet\\ & \backslash & & / & \\ & & \bullet & & \end{array}\right\}$, $in(IV,2) = \sqrt[3]{2}$;

$(3) = \left\{\begin{array}{ccccc}\bullet & & & & \bullet\\ & \backslash & & / & \\ & & \bullet & & \\ & & | & & \\ & & \bullet & & \end{array}\right\}$, $(4) = \left\{\begin{array}{ccccc} & & \bullet & & \\ & & | & & \\ & & \bullet & & \\ & / & & \backslash & \\ \bullet & & & & \bullet\end{array}\right\}$, $in\,(IV,3) = in(IV,4) = \sqrt[3]{2}$;

$(5) = \left\{\begin{array}{ccc} & & \bullet\\ & & |\\ \bullet & & \bullet\\ & \backslash & |\\ & & \bullet\end{array}\right\}$, $(6) = \left\{\begin{array}{ccc} & & \bullet\\ & / & |\\ \bullet & & \bullet\\ & & |\\ & & \bullet\end{array}\right\}$; $\chi_{5,6}(x) = x(x^3 - x - 1)$ and

$1.32 < in\,(IV,5) = in\,(IV,6) < 1.33$;

$(7) = \left\{\begin{array}{cc}\bullet & \bullet \\ | & | \\ \bullet & \bullet\end{array}\right\}$, $in\,(IV,7) = \sqrt{2}$; $(8) = \left\{\begin{array}{cc} & \bullet \\ & | \\ & \bullet \\ & | \\ \bullet & \bullet\end{array}\right\}$, $\chi_8(x) = x(x^3 - x^2 - 1)$

and

$1.46 < in\,(IV,8) < 1.47$;

$(9) = \left\{\begin{array}{ccc}\bullet & & \bullet \\ | & \diagdown & | \\ \bullet & & \bullet\end{array}\right\}$, $in\,(IV,9) = \sqrt{3}$;

$(10) = \left\{\begin{array}{ccccc} & & \bullet & & \\ & / & | & \diagdown & \\ \bullet & & \bullet & & \bullet\end{array}\right\}$, $(11) = \left\{\begin{array}{ccccc}\bullet & & \bullet & & \bullet \\ & \diagdown & | & / & \\ & & \bullet & & \end{array}\right\}$, $in(10) = in(11) = \sqrt{3}$;

$(12) = \left\{\begin{array}{cccccc} & & & \bullet & & \\ & & / & & \diagdown & \\ \bullet & \bullet & & & & \bullet\end{array}\right\}$, $(13) = \left\{\begin{array}{cccccc}\bullet & \bullet & & & & \bullet \\ & & \diagdown & & / & \\ & & & \bullet & & \end{array}\right\}$,

$in\,(IV,12) = in\,(IV,13) = 2$;

$(14) = \left\{\begin{array}{ccc}\bullet & & \bullet \\ | & \times & | \\ \bullet & & \bullet\end{array}\right\}$, $in(IV,14) = 2$;

$(15) = \left\{\begin{array}{ccc} & & \bullet \\ & & | \\ \bullet & \bullet & \bullet\end{array}\right\}$, $\chi_{15}(x) = x^2(x^2 - 2x - 1)$ and $in\,(IV,15) = 1 + \sqrt{2}$;

$(16) = \{\bullet\ \bullet\ \bullet\ \bullet\}$, $in\,(IV,16) = 4$.

Note that posets $(IV,2)$, $(IV,3)$ and $(IV,4)$ are Q-equivalent. For posets $N = (IV,9)$ and $F_4 = (IV,11)$ we have $in\,N = in\,F_4 = \sqrt{3}$, but N and F_4 are non-Q-equivalent.

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V.V. Kirichenko, V.N. Zhuravlev
Department of Mechanics and Mathematics
Kiev National Taras Shevchenko University
Vladimirskaya 64, Kiev
01033 Ukraine
E-mail:*vkir @univ.kiev.ua*

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A.V. Zelensky
Department of Mathematics Chernivcy
National Yurii Fedkovich University
Universytetska 28, Chernivcy
58012 Ukraina