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Transfer properties in radical theory

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Abstract. A functor is said to reflect radical classes if under this functor the inverse image of a radical class is always a radical class.Prototypical examples of such functors include polynomial and matrix functors and various forgetful functors.This paper is for the most part a survey of known results concerning radical reflections,but there are a few new results,including a generalization to right alternative rings of a well known result of Andrunakievici on upper radicals of simple associative rings.

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A functor $\phi\colon \mathcal{C} \to \mathcal{D}$ is said to *reflect radical classes* if for every radical class $\mathcal{R}$ in $\mathcal{D}$, the class $\mathcal{R}^* = \phi^{-1}(\mathcal{R}) = \{\mathcal{A}\colon \phi(\mathcal{A}) \in \mathcal{R}\}$ is a radical class in $\mathcal{C}$. This notion was studied systematically in the '70s, but there are many examples in the earlier and later literature, and the concept has been investigated by (in no particular order, and with apologies to those overlooked) Amitsur, Ortiz, Gardner, Stewart, Puczyłowski, Sierpińska, Beattie, Fang, Krempa, Skosyrskii, Widarma, Thedy, McCrimmon, Arnautov, Vodinchar, Slin'ko and Soweiter. (This joke is due to Georges Perec.) From the number of talks at the Chişinău conference which mentioned problems, questions and results which concern examples of radical reflections, it seems that the idea has considerable contemporary relevance for radical theorists.

There are a number of significant ways in which the study of radical reflections (and other methods for transferring radicals from one context to another) can contribute to radical theory.

• As a source of examples.

• By describing interactions between radicals and algebraic constructions (matrix rings, polynomial rings and so on).

• By generalizing particular radicals to new settings (e.g. finding the "correct version" of local nilpotence for varieties of non-associative rings).

• By extending known results concerning radicals in one context to analogous results in another (e.g. existence of hereditary semi-simple classes, lattice properties).

• By transferring a "traditional" radical theory to a non-standard setting, perhaps comparing the transferred theory with some ad hoc version of radical theory set up in the latter.

• By transferring some kind of radical theory to a context where no obvious one exists (as when a category "suitable for radical theory" is equivalent to an "unsuitable" one and an equivalence effects the transfer).

In what follows we shall give examples to illustrate all of these possibilities. While on the whole we are presenting a survey of known results, there are a few novelties. We do not work at a fixed level of generality. For much of the time we work with multioperator groups for the sake of definiteness. Laci Márki suggested that semi-abelian categories in the sense of [1] might provide an appropriate context. Certainly the group-based structures of [2] are sometimes too general (see Example 1.6). On the whole our terminology is consistent with [2] and [3]. In some of the examples, categories are given self-explanatory bold-faced names, but on occasion they are also referred to more informally.

A preliminary version of part of this paper was contained, together with some other topics, in a talk to the Pat Stewart Memorial Session of the 2002 APICS mathematics meeting in Sackville, New Brunswick.

1 Reflected Radicals

Let $\mathcal{C}$ and $\mathcal{D}$ be varieties of multioperator groups. We say that a functor $\phi\colon \mathcal{C} \to \mathcal{D}$ *reflects radical classes* if for every radical class $\mathcal{R}$ in $\mathcal{D}$, the class $\mathcal{R}^* = \phi^{-1}(\mathcal{R})$ is a radical class in $\mathcal{C}$.

Theorem 1.1. *(See [4]). If ϕ is exact and preserves unions of chains of normal subobjects, then ϕ reflects radical classes.*

We list some examples of functors satisfying the conditions of 1.1. In each case the action of the given functor on morphisms is well known.

Example 1.2.

(i)**Rings** $\to$ **Rings**; $A \mapsto A[X]$.

(ii)**Rings** $\to$ **Rings**; $A \mapsto [S]$ (semigroup ring; fixed semigroup S).

(iii)**Rings** $\to$ **Rings**; $A \mapsto M_n(A)$(matrix ring, fixed n).

(iv)**Rings** $\to$ **Jordan Rings**; $(A, +, \cdot) \mapsto (A, +, \odot)$ where $a \odot b = ab + ba$.

(v)**Rings** $\to$ **Lie Rings**;$(A, +, \cdot) \mapsto (A, +, [*, *])$.

(vi)**Rings** $\to$ **Abelian Groups**; $(A, +, \cdot) \mapsto (A, +)$.

(vii)$K-$**Algebras** $\to$ **Rings** (for a commutative ring K with identity); forgetful functor.

(viii)**Rings** $\to$ **Rings**; $A \mapsto A^{op}$ (opposite ring).

(ix)**Differential Rings** $\to$ **Rings**; forgetful functor.

(x)**Rings with Involution** $\to$ **Rings**; forgetful functor.

(xi)**Quasiregular Rings** $\to$ **Groups**; $(A+, \cdot) \mapsto (A, \circ)$.

We shall see later that neither of the conditions of 1.1 is necessary for the reflection of radical classes, but as the following few examples show, neither is sufficient either.

Example 1.3. (See [4].) The functor from **Rings** to **Rings** which associates with each ring A the power series ring $A[[X]]$ is exact but does not reflect radical classes.(For some information on radicals and power series, see [5].)

Example 1.4. The functor ϕ: **Rings** $\to$ **Rings**, where $\phi(A) = A^2$ for each A and ϕ acts on homomorphisms by restriction, is not exact: if

$$0 \to I \to A \to A/I \to 0$$

is exact, then so is

$$0 \to I \cap A^2 \to A^2 \to A^2/I \cap A^2 \cong (A^2 + I)/I = (A/I)^2 \to 0$$

but in general I^2 and $I \cap A^2$ can be quite different. For instance, if I is a ring with $I^2 = 0$ and $A = I * \mathbf{Z}$ is the standard unital extension, then $I^2 = 0$ and $I \cap A^2 = I \cap A = I$. However, as one shows easily, ϕ preserves unions of chains of ideals. Let $\mathcal{R}$ be the (radical)class of boolean rings. If R is a ring with $R^3 = 0 \neq R^2$, then trivially$(R^2)^2 \in \mathcal{R}$ and $(R/R^2)^2 = 0 \in \mathcal{R}$ so R^2 and $R/R^2 \in \mathcal{R}^*$. But $R \notin \mathcal{R}^*$ as $0 \neq R^2 \notin \mathcal{R}$. Thus $\mathcal{R}^*$ is not a radical class. (Note that ϕ preserves quotients.)

Example 1.5. Let ϕ: **Abelian Groups** $\to$ **Abelian Groups** assign the *socle* and act on homomorphisms in the usual way. If

$$0 \to H \to G \to G/H \to 0$$

is exact, then so is

$$0 \to \phi(H) \to \phi(G) \to \phi(G)/\phi(H) \to 0$$

but if, e.g., $H = \mathbf{Z}(p)$ and $G = \mathbf{Z}(p^\infty)$, then $\phi(G)/\phi(H) = 0$, while $\phi(G/H) \cong \phi(\mathbf{Z}(p^\infty) \cong \mathbf{Z}(p)$. On the other hand, if $\{H_\lambda : \lambda \in \Lambda\}$ is a chain of subgroups of G, then

$$\phi(\bigcup_{\lambda\in\Lambda} H_\lambda) = \{x \in \bigcup_{\lambda\in\Lambda} H_\lambda : 0(x)\text{is square-free}\} = \bigcup_{\lambda\in\Lambda} \phi(H_\lambda).$$

Let $\mathcal{T}_p$ be the (radical) class of abelian p-groups, q a prime $\neq p$. Then $\phi(\mathbf{Z}) = 0 \in \mathcal{T}_p$ but $\phi(\mathbf{Z}/q\mathbf{Z}) = \mathbf{Z}/q\mathbf{Z} \notin \mathcal{T}_p$, so $\mathbf{Z} \in \mathcal{T}_p^*$ while $\mathbf{Z}/q\mathbf{Z} \notin \mathcal{T}_p^*$. Hence ϕ does not reflect radical classes. (Note that ϕ takes subgroups to subgroups.)

Though we shall not seriously address the problem of characterizing the functors which reflect radical classes, we note one further pertinent example of one which doesn't. One of our categories is not a variety of multioperators here, but the functor is a forgetful one and provides some contrast with some of our cited examples in 1.2.

Example 1.6. Let ϕ be the forgetful functor from **Hausdorff Topological Groups** to **Abelian Groups** (forget the topology). Let $\mathcal{R}$ be a radical class of abelian groups, $\{A_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{R}$. Give each A_λ the discrete topology and let P denote the cartesian product of the A_λ with the product topology. Then $\bigoplus_{\lambda\in\Lambda} A_\lambda$ with the subspace topology from P is in $\mathcal{R}^*$, so if $\mathcal{R}^*$ is a radical class, $\mathcal{R}^*(P)$ is a closed subgroup containing $\bigoplus_{\lambda\in\Lambda} A_\lambda$. But the latter is dense, so we must have $P \in \mathcal{R}^*$. Hence if $\mathcal{R}^*$ is a radical class, then $\mathcal{R}$ must be closed under direct products.

Closure under direct products is not enough, however. For a prime p, let $\mathbf{Q}(p)$ be the group $\{m/p^n : m, n \in \mathbf{Z}\}$. Let $\mathcal{D}_p$ be the (radical) class of p-divisible groups. Let H be a torsion-free group of rank 2 with $\mathcal{D}_p(H) \cong \mathbf{Q}(p)$ and $H/\mathcal{D}_p(H) \cong \mathbf{Q}(q)$ for a prime $q \neq p$. Then H has no elements of infinite q-height and $\mathcal{D}_p(H)$ is dense in H for the q-adic topology. If $\mathcal{D}_p^*$ were a radical class it would have to contain H. But $H \notin \mathcal{D}_p$.

The analogous question for hausdorff topological *rings* (*algebraic radicals*) has been treated in [6].

2 The Local Effect

Reflection of radical classes as thus far described is a global phenomenon. There is also a local phenomenon, which we can illustrate by first observing that for some ring radical classes $\mathcal{R}$ we have $\mathcal{R}(A[X]) = \mathcal{R}(A)[X]$ for all A, and then asking, if this equation is not universally valid but for *some* ring A we have $\mathcal{R}(A[X]) = I[X]$ for *some* $I \triangleleft A$, what is the nature of I?

We maintain the notation and assumptions of the previous section.

Lemma 2.1. *Let ϕ satisfy the conditions of 1.1. If $\mathcal{R}$ is a radical class in $\mathcal{D}$ then $\phi(\mathcal{R}^*(A)) \subseteq \mathcal{R}(\phi(\mathcal{A}))$ for each $A \in \mathcal{C}$.*

Proof Since $\mathcal{R}^*(A) \in \mathcal{R}^*$ we have $\phi(\mathcal{R}^*(A)) \in \mathcal{R}$. But
$\mathcal{R}^*(A) \triangleleft A$, so $\phi(\mathcal{R}^*(A)) \triangleleft \phi(A)$.

Theorem 2.2. *For $\phi\colon \mathcal{C} \to \mathcal{D}$ as in Theorem 1.1, the following are equivalent for $A \in \mathcal{C}$. (i)$\mathcal{R}(\phi(A)) = \phi(\mathcal{R}(A))$; (ii)$\mathcal{R}(\phi(A)) = \phi(I)$ for some $I \triangleleft A$.*

Proof (ii)⇒(i):If $\mathcal{R}(\phi(A)) = \phi(I)$,where $I \triangleleft A$, then $\phi(I) \in \mathcal{R}$, so $I \in \mathcal{R}^*$ and hence $I \subseteq \mathcal{R}^*(A)$. But then $I \triangleleft \mathcal{R}^*(A)$ so

$$\mathcal{R}(\phi(A)) = \phi(I) \triangleleft \phi(\mathcal{R}^*(A)).$$

The reverse inclusion follows from 2.1.

Corollary 2.3. *For a ring A and a radical class $\mathcal{R}$ of rings, $\mathcal{R}(A[X]) = I[X]$ for some $I \triangleleft A$ if and only if $\mathcal{R}(A[X]) = \mathcal{R}^*(A)[X]$.*

Example 2.4. (See [7].) Let $\mathcal{C} = \mathcal{D} =$the category of rings, $\phi(A) = M_n(A)$ for all A. Then for every A and every $\mathcal{R}$ there is an ideal I of A for which $\mathcal{R}(M_n(A)) = M_n(I)$. Thus

$$\mathcal{R}(M_n(A)) = \mathcal{R}(\phi(A)) = \phi(\mathcal{R}^*(A)) = M_n(\mathcal{R}^*(A)).$$

Example 2.5. (See [8].) Let ϕ be the forgetful functor from rings to abelian groups. If $\mathcal{R}$ is any radical class of abelian groups, then for every G, $\mathcal{R}(G)$ is a fully invariant subgroup of G. Hence for a ring A, $\mathcal{R}((A, +))$ is a fully invariant subgroup of $(A, +)$. Since left and right multiplications are additive endomorphisms, $\mathcal{R}(A, +)$ is an ideal of A, or, more precisely, $\mathcal{R}(A, +) = (I, +)$ for some $I \triangleleft A$. this I must be $\mathcal{R}^*(A)$. Thus $\mathcal{R}(A, +) = (\mathcal{R}^*(A), +)$ for all $\mathcal{R}$, A.

Example 2.6. (See [9].) Let ϕ be the forgetful functor from algebras over a field K to rings. For every radical class $\mathcal{R}$ of rings, $\mathcal{R}(A)$ is an algebra ideal of every K-algebra A. Thus "$\mathcal{R}(A) = \mathcal{R}^*(A)$".

Example 2.7. (See [10].) For the functor ϕ from right alternative algebras over $\mathbf{Q}(2)$ to Jordan algebras over $\mathbf{Q}(2)$ where the multiplication is replaced by $a \odot b = \frac{1}{2}(ab + ba)$, if $\mathcal{R}$ is a *non-degenerate* radical of Jordan algebras (semi-simple algebras have no strong zero- divisors) then the same is true of $\mathcal{R}^*$, and $(\mathcal{R}^*(A), +, \odot) = \mathcal{R}(A, +, \odot)$, since $\mathcal{R}(\phi(A)) = \mathcal{R}(A, +, \odot)$ is $(I, +, \odot)$ for an ideal I of A (for every A). This result is used in [10] to transfer many standard radicals from Jordan to right alternative algebras and show that they retain significant properties. (The same functor can be used to define a substitute for local nilpotence in right alternative algebras [11].) Analogous results for some other types of algebras are given in [12]. On the other hand, the similar notion of reflection of radicals from *Lie* algebras seems not to have attracted much attention.

The conditions of 2.2 are met (for a given A and $\mathcal{R}$) when $\mathcal{R}(A)$ is "highly invariant", maintaining normality when a richer, or at least different structure is imposed (as in the passage from abelian groups to rings or from right alternative to Jordan rings). In the cases of the polynomial and matrix functors, the conditions correspond to "well-behaved ideals"; e.g. ideals of matrix rings which have to be matrix rings over ideals. This piece of unification is perhaps of some independent interest.

3 Properties Preserved by the Lower Radical Construction

We consider a functor ϕ as in Section 1, but with $\mathcal{C} = \mathcal{D}$, and call a class $\mathcal{K} \subseteq \mathcal{C}$ a *ϕ-invariant class* if $\phi(A) \in \mathcal{K}$ for all $A \in \mathcal{K}$. We can then ask whether ϕ-invariance is preserved by the lower radical construction in the sense indicated in the following result.

Proposition 3.1. *Let $\phi\colon \mathcal{C} \to \mathcal{C}$ satisfy the conditions of Section 1. Let $\mathcal{M}$ be a homomorphically closed subclass of $\mathcal{C}$, $L(\mathcal{M})$ its lower radical class. If $\mathcal{M}$ is ϕ-invariant, then $L(\mathcal{M}) \subseteq L(\mathcal{M})^*$ and $L(\mathcal{M})$ is ϕ-invariant.*

Proof If $A \in \mathcal{M}$, then $\phi(A) \in \mathcal{M} \subseteq L(\mathcal{M})$ so $A \in L(\mathcal{M})^*$. Thus $\mathcal{M} \subseteq L(\mathcal{M})^*$ so $L(\mathcal{M}) \subseteq L(\mathcal{M})^*$. Hence for all $B \in L(\mathcal{M})$ we have $B \in L(\mathcal{M})^*$, i.e. $\phi(B) \subseteq L(\mathcal{M})$.

Thus, e.g., if a class of (associative) rings is closed under formation of polynomial rings, then so is its lower radical class ([13]; see [14] for the corresponding result for Jordan rings). Likewise a class closed under formation of $n \times n$ matrix rings forms a lower radical class with the same property.

The following property is also worth looking at.

$$A \in \mathcal{M} \text{ and } \phi(A) \cong \phi(B) \Rightarrow B \in \mathcal{M} - - - - - - - - (\dagger)$$

When is (†) preserved under the lower radical construction? We have just a little information about this.

Proposition 3.2. *If ϕ is a monoid ring functor ($\phi(A) = A[S]$, S fixed) from rings to rings, and if $\mathcal{M}$ is homomorphically closed and ϕ-invariant, then $\mathcal{M}$ satisfies* (†).

Proof If $A \in \mathcal{M}$ and $A[S] \cong B[S]$, then by ϕ-invariance, $A[S] \in \mathcal{M}$, so $B[S] \in \mathcal{M}$ and thus $B \in \mathcal{M}$.

Using 3.1 and 3.2, we get

Corollary 3.3. *If $\mathcal{M}$ is homomorphically closed and ϕ-invariant and satisfies* (†), *then $L(\mathcal{M})$ satisfies* (†).

Example 3.4. When ϕ is the forgetful functor from rings to abelian groups, (†) need not be preserved under the lower radical construction. For instance $\{GF(p), \mathbf{Z}(p)^0\}$ satisfies (†) but its lower radical class excludes $GF(p^2)$, though this field has the same additive group as $GF(p) \oplus \mathbf{Z}(p)^0$.

4 Categorical Equivalence

If $\mathcal{C}$ and $\mathcal{D}$ are varieties of multioperator groups, and $\phi\colon \mathcal{C} \to \mathcal{D}$ is an equivalence, with $\psi\colon \mathcal{D} \to \mathcal{C}$ the complementary equivalence, then for a radical class $\mathcal{R}$ in $\mathcal{D}$ we denote $\phi^{-1}(\mathcal{R})$ by$\mathcal{R}^*$ as before, and for a radical class $\mathcal{U}$ in $\mathcal{C}$ we let $\psi^{-1}(\mathcal{U}) = \mathcal{U}^{\#}$. As ϕ and ψ preserve limits and colimits, $\mathcal{R}^*$ and $\mathcal{U}^{\#}$ are always radical classes. Now

$$\mathcal{R}^{*\#} = \{D \in \mathcal{D} : \psi(D) \in \mathcal{R}^*\} = \{D \in \mathcal{D} : \phi\psi(D) \in \mathcal{R}\},$$

so, since $D \cong \phi\psi(D)$ we have $\mathcal{R}^{*\#} = \mathcal{R}$ for every radical class $\mathcal{R}$ in $\mathcal{D}$, and similarly $\mathcal{U}^{\#*} = \mathcal{U}$ for every radical class $\mathcal{U}$ in $\mathcal{C}$. Thus we have

Proposition 4.1. *A categorical equivalence ϕ between varieties $\mathcal{C}$ and $\mathcal{D}$ of multioperator groups induces a bijection $\mathcal{R} \leftrightarrow \mathcal{R}^*$ between radical classes in $\mathcal{D}$ and $\mathcal{C}$.*

It is easy to see that 4.1 has no converse; if F is a finite field and K an infinite field, then in the categories of $F-$ and $K-$ vector spaces there are only the trivial radical classes, but the categories are not equivalent since all pairs of non-zero K-vector spaces have infinite Hom-sets but this is not so for F-vector spaces.

One feels that equivalent categories (of multioperator groups or not) should be "radically the same". It is possible for a category which supports some kind of radical theory to be equivalent to one which does not (at least not in any obvious sense). In such circumstances it seems reasonable to use the equivalence to *induce* radical notions in the second category. If there is already some kind of radical theory in the second category, a comparison of the two competing versions may prove instructive. For instance radical theory for modules over a ring R can be transferred easily to the category of *affine* R-modules [15], or that of *pointed* R-modules. The categories of affine and pointed modules over certain rings are equivalent to certain categories of quasigroups [16],[17]. We shall consider some of these module-quasigroup connections elsewhere. In the case of *idempotent* quasigroups, there is already a version of radical theory [18], which contributes a further strand to this story. There are also equivalences between categories of MV-algebras and l-rings and abelian l-groups [19],[20],[21]. The l-structures are of course multioperator groups, but MV-algebras are rather different. Would radical theory reflected to MV-algebras by these equivalences produce anything interesting?

5 Transferring Radicals from a Subvariety

A rather different kind of functor from those treated hitherto enables us to reflect radical classes to a variety from a subvariety. If the radical theory of the subvariety is well understood, this technique may provide useful information about radicals in the larger variety. We shall again work with multioperator groups.

For a subvariety $\mathcal{V}$ of a variety $\mathcal{W}$, for each $A \in \mathcal{W}$ we let

$$A(\mathcal{V}) = \bigcap\{I : I \lhd A, A/I \in \mathcal{V}\}.$$

If $f\colon A \to B$ is a homomorphism in $\mathcal{W}$ and B/J is in $\mathcal{V}$, then denoting the natural map $B \to B/J$ by p, we have

$$A/Ker(pf) \cong Im(pf) = (Im(f) + J)/J \subseteq B/J \in \mathcal{V},$$

so $A(\mathcal{V}) \subseteq Ker(pf)$ and so $f(A(\mathcal{V})) \subseteq Ker(pf) = J$. This being so for all such J, we have $f(A(\mathcal{V})) \subseteq B(\mathcal{V})$. Thus the correspondence $A \mapsto A(\mathcal{V})$ defines a functor (*subfunctor of the identity*). Now (for f as above) we get a homomorphism $\hat{f}\colon A/A(\mathcal{V}) \to B/B(\mathcal{V})$ by defining $\hat{f}(a + A(\mathcal{V})) = f(a) + B(\mathcal{V})$ for each $a \in A$. This makes a functor of the correspondence $A \mapsto A/A(\mathcal{V})$ (*factor functor of the identity*) and this is the functor we shall use.

We shall denote by $U_{\mathcal{V}}(), U_{\mathcal{W}}()$ the upper radical in $\mathcal{V}, \mathcal{W}$ respectively.

Theorem 5.1. *(See [22].) Let $\mathcal{V}$ be a subvariety of $\mathcal{W}$. For a radical class $\mathcal{R}$ in $\mathcal{V}$ let $\mathcal{R}^* = \{A \in \mathcal{W} : A/A(\mathcal{V}) \in \mathcal{R}\}$. Then if $\mathcal{R}$ has semi-simple class $\mathcal{S}$, we have $\mathcal{R}^* = U_{\mathcal{W}}(\mathcal{S})$. In particular, $\mathcal{R}^*$ is a radical class in $\mathcal{W}$.*

(We note that no matter what $\mathcal{W}$ is, $U_{\mathcal{W}}((\mathcal{S})$ exists, as $\mathcal{S}$ is a regular class in both $\mathcal{V}$ and $\mathcal{W}$.)

The transfer obtained is likely to be useful only if the classes $\mathcal{R}^*$ are not too big. For instance if $\mathcal{W}$ is the variety of alternative rings and $\mathcal{V}$ that of associative rings, there are lots of Cayley-Dickson rings which must belong to every $\mathcal{R}^*$.We impose another condition and get a stronger conclusion than that of 5.1 which is useful.

Theorem 5.2. *(See [22].) Let $\mathcal{V}, \mathcal{W}$ be as in 5.1 and suppose further that $A(\mathcal{V})/A(\mathcal{V})(\mathcal{V}) \in \mathcal{R}$ (i.e. $A(\mathcal{V}) \in \mathcal{R}^*$) for every $A \in \mathcal{W}$. Then*

(i)$\mathcal{R}^*(A)/A(\mathcal{V}) = \mathcal{R}(A/A(\mathcal{V}))$ for all $A \in \mathcal{V}$ and

(ii)$\mathcal{S}$ is the semi-simple class of $\mathcal{R}^*$.

Here we have some possibility of extending a result concerning well-behaved radicals from $\mathcal{V}$ to $\mathcal{W}$, since some semi-simple classes in $\mathcal{V}$ remain semi-simple classes in $\mathcal{W}$, and there are properties of semi-simple classes which make for well-behaved radicals. We give some illustrations of the situation described in 5.2.

Example 5.3. $\mathcal{V}$ is a semi-simple radical class in $\mathcal{W}$ if and only if $A(\mathcal{V}) = A(\mathcal{V})(\mathcal{V})$ for every $A \in \mathcal{W}$ [23], i.e. $A(\mathcal{V}) \in \{0\}^* = U_{\mathcal{W}}(\mathcal{V})$ for every A. If $\mathcal{R}$ is any radical class in $\mathcal{V}$, then each $A(\mathcal{V})$ is in $\mathcal{R}^*$ and the semi-simple class of $\mathcal{R}$ in $\mathcal{V}$ remains a semi-simple class in $\mathcal{W}$.

Example 5.4. If $\mathcal{W}$ is the class of all (not necessarily associative) rings, $\mathcal{V}$ the class of associative rings, then in $\mathcal{W}$ the only hereditary semi-simple classes are those corresponding to A-radicals [24] while all semi-simple classes in $\mathcal{V}$ are hereditary. Hence only A-radicals (in $\mathcal{W}$) satisfy the hypotheses of 5.2.

It is more convenient to have examples of the phenomenon in 5.2 where our starting point is a radical class in $\mathcal{W}$ rather than in $\mathcal{V}$.

Proposition 5.5. ([22])(Notation as in 5.2.) *If $\mathcal{U}$ is a radical class in $\mathcal{W}$ and $A(\mathcal{V}) \in \mathcal{U}$ for all $A \in \mathcal{W}$, then $\mathcal{U} = (\mathcal{U} \cap \mathcal{V})^*$ and $\mathcal{U}$, $\mathcal{U} \cap \mathcal{V}$ are related as $\mathcal{R}^*$ and $\mathcal{R}$ are related in (i),(ii) of 5.2.*

Example 5.6. We illustrate 5.5 by considering the case where $\mathcal{W}$ is the class of *right alternative rings*, $\mathcal{V}$ the class of *alternative rings*. By a result of Skosyrskii [25] $A(\mathcal{V})$, which is called the *alternator* of A, is contained in the McCrimmon radical of A (cf. 2.7). The McCrimmon radical is the upper radical defined by the class of *non-degenerate rings*, i.e. rings with no strong zero-divisors. For this example, "ring" always means "ring in which division by 2 is possible"; in particular, characteristic 2 is avoided. Thus if $\mathcal{U}$ is any non-degenerate radical class in $\mathcal{W}$ (i.e. all $\mathcal{U}$-semi-simple rings are non-degenerate) then $A(\mathcal{V}) \in \mathcal{U}$. Hence, by 5.5, $\mathcal{U}$ (in $\mathcal{W}$) and $\mathcal{U} \cap \mathcal{V}$ (in $\mathcal{V}$) have the same semi-simple class. In particular, non-degenerate radicals of right alternative rings have hereditary semi-simple classes.

This can be improved.

Theorem 5.7. *(See [22].) Let $\mathcal{W}$ be a variety, $\mathcal{V}$ a subvariety, $\mathcal{U}$ a hereditary radical class in $\mathcal{W}$ such that $A(\mathcal{V}) \in \mathcal{U}$ for all $A \in \mathcal{W}$. If every radical class in $\mathcal{V}$ satisfies* ADS *then every radical class $\mathcal{T}$ in $\mathcal{W}$ with $\mathcal{U} \subseteq \mathcal{T}$ also satisfies* ADS.

Now all radical classes of alternative rings satisfy ADS so we have

Corollary 5.8. *(See [22].) Every non-degenerate radical class of right alternative rings satisfies* ADS.

We conclude with a more detailed result obtained similarly.

Theorem 5.9. *Let $\mathcal{M}$ be a class of simple right alternative rings, $\mathcal{U}$ the upper radical class defined by $\mathcal{M}$ (in the class of right alternative rings). The following conditions are equivalent.*

(i) $\mathcal{U}$ is hereditary and has the intersection property with respect to $\mathcal{M}$.
(ii) All rings in $\mathcal{M}$ are unital.

(This theorem was proved for associative rings by Andrunakievich [26] and can be generalized to alternative rings by means of results of Suliński [27]. More recently, Leavitt [28] has shown that for associative rings (ii) is equivalent to the intersection property alone, and taking account of the fact that non-unital simple alternative rings are associative, one can show straightforwardly that the stronger result is valid in the alternative case too.)

Proof $\neg$ (ii)$\Rightarrow \neg$ (i). If $\mathcal{M}$ contains a non-unital ring S, let S^* be the ring obtained form S by the adjunction of the identity of $\mathbf{Q}$ or $\mathbf{Z}_p$ to match the characteristic of S (so that S^*/S is isomorphic to the appropriate field). The only simple image of S^* is S^*/S. If $S^*/S \notin \mathcal{M}$, then $S^* \in \mathcal{U}$ but $S \notin \mathcal{U}$, so $\mathcal{U}$ is not hereditary. If $S^*/S \in \mathcal{M}$, then S^* is in the semi-simple class of $\mathcal{U}$ but is subdirectly irreducible and non-simple, so that $\mathcal{U}$ does not have the intersection property with respect to $\mathcal{M}$. (This is a familiar argument in the associative case.)

(ii)$\Rightarrow$(i). Suppose all rings in $\mathcal{M}$ are unital. As every right alternative ring has nil alternator, the rings in $\mathcal{M}$ are alternative. Let $\mathcal{S}$ be the class of subdirect products of rings in $\mathcal{M}$. Then $\mathcal{S}$ is a semi-simple class in the universal class of alternative rings and $\mathcal{U}$ is its upper radical class in the universal class of *right* alternative rings. Since the alternator of every ring is in $\mathcal{U}$, 5.5 says that $\mathcal{S}$ is the semi-simple class of $\mathcal{U}$ (in the class of right alternative rings). Hence $\mathcal{U}$ has the intersection property with respect to $\mathcal{M}$.

Now a radical class with ADS is hereditary if and only if its semi-simple class is closed under essential extensions. (This is proved as for the associative case in [29].)Every radical class of alternative rings has ADS, so $\mathcal{S}$ is closed under *alternative* essential extensions. If $A \in \mathcal{S}$, $A \triangleleft^{\bullet} B$ and B is right alternative, let J be the alternator of B. Then $J \cap A$ is a nil ideal of A and a member of $\mathcal{S}$, so $J \cap A = 0$, whence $J = 0$ and B is alternative. But then B is in $\mathcal{S}$. Thus $\mathcal{S}$ is closed under right alternative essential extensions, so that by 5.8, $\mathcal{U}$ is hereditary.

It's well known that 5.9 is not valid for the class of all (not necessarily associative) rings, and it would be interesting to know how far beyond right alternative rings it extends. It does not extend to power-associative rings. The following example was used by Henriksen [30] for other purposes.

Example 5.10. Let F be a field, and let R be an F-algebra with basis $\{a, b, e\}$, $ab = e = -ba, e^2 = e$ and all other basis products zero. If $\alpha, \beta, \gamma \in F$ then $(\alpha a + \beta b + \gamma e)^2 = \gamma^2 e$, $(\alpha a + \beta b + \gamma e)\gamma^2 e = \gamma^3 e = \gamma^2 e(\alpha a + \beta b + \gamma e)$ and so on, so R is power-associative. If $g\colon R \to F$ is a homomorphism, then $g(a)^2 = 0 = g(b)^2$ so $g(a) = 0 = g(b)$, and then $g(e) = g(ab) = g(a)g(b) = 0$, so $g = 0$. Thus R is in the upper radical class defined by $\{F\}$, but $F \cong Fe \triangleleft R$, so the upper radical class is not hereditary.

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