BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(44), 2004, Pages 40–45
ISSN 1024–7696

Rings over which some preradicals are torsions

I.D. Bunu

Abstract. Let R be an associative ring with identity and z be a pretorsion such that its filter consists of the essential left ideals of the ring R. In this paper, it is proved that every preradical $r \geq z$ of $R - Mod$ is a torsion if and only if the ring R is a finite direct sum of pseudoinjective simple rings.

Mathematics subject classification: 16S90.
Keywords and phrases: Torsion (pretorsion), essential ideal, strongly prime ring, pseudoinjective module (ring).

Let z be the Goldie pretorsion of $R - Mod$ category of left $R-$ modules over the associative ring R with identity, i.e. its filter consists of essential left ideals of this ring.

In this paper some rings are described, over which all preradicals $r \geq z$ are torsions. It is proved that such rings are exactly those that can be decomposed in a finite direct sum of pseudoinjective simple rings.

First of all we present some preliminary notions and definitions.

01. A *preradical* r of $R - Mod$ is a subfunctor of the identity functor of $R - Mod$ [1, 2].

A preradical r is called

– *radical* if $r\,(M/r\,(M)) = 0$ for any $M \in R - Mod$;

– *pretorsion* if $r\,(N) = N \cap r\,(M)$ for any submodule N of an arbitrary module M;

– *torsion* if r is a radical and pretorsion.

02. An arbitrary preradical r of category $R - Mod$ defines two classes of modules: $\mathcal{R}\,(r) = \{M \in R - Mod \mid r\,(M) = M\}$ and $\mathcal{P}\,(r) = \{M \in R - Mod \mid r\,(M) = 0\}$. Modules of the class $\mathcal{R}\,(r)$ are called *r-torsion*, and of the class $\mathcal{P}\,(r)$ are called r-torsion free.

Preradicals 0 and ε for which $\mathcal{P}\,(0) = R - Mod$ and $\mathcal{R}\,(\varepsilon) = R - Mod$ are called *nul* and *identity,* respectively.

03. If r_1 and r_2 are two arbitrary preradicals, then $r_1 \leq r_2$ means that $r_1\,(M) \subseteq r_2\,(M)$ for any $M \in R - Mod$.

The intersection of preradicals r_1 and r_2 is the preradical $r_1 \wedge r_2$ determined by the rule: $(r_1 \wedge r_2)\,(M) = r_1\,(M) \cap r_2\,(M)$ for any $M \in R - Mod$.

The sum of preradicals r_1 and r_2 is the preradical $r_1 + r_2$ defined by the relation $(r_1 + r_2)\,(M) = r_1\,(M) + r_2\,(M)$ for any $M \in R - Mod$.

04. The least pretorsion containing preradical r is denoted by $h(r)$. It always exists and is determined by the equality $h(r)(M) = M \cap r\left(\hat{M}\right)$, where $\hat{M}$ is the injective envelope of an arbitrary module $M \in R - Mod$ ([1], p.23).

For any pretorsion r the least torsion $\bar{r}$ containing it exists and satisfies the property $r(M) \subseteq' \bar{r}(M)$ for any $M \in R - Mod$, ([1], 1.8 item Cyrillic “b”).

05. Every nonzero module M determines the radical r_M such that $r_M(A) = \cap Kerf$ for all homomorphism $f \in Hom_R(A, M)$ for every $A \in R - Mod$. The radical r_M is the greatest among all preradicals r with the property $r(M) = 0$ ([1], p.13). If the module M is injective, then the radical r_M is a torsion ([1], p.32). Moreover $r_M(R) = (0 : M)$.

06. A module M is called *pseudoinjective* if for any monomorphism $i : B \to A$ and every homomorphism $f : B \to M$ there are such homomorphisms $\alpha : M \to M$ and $\bar{f} : A \to M$ that $0 \neq \alpha f = i \bar{f}$.

The following conditions are equivalent:

(1) M is a pseudoinjective module.

(2) $r_M = r_{\hat{M}}$.

(3) The radical r_M is a torsion ([3], p.45).

07. The Goldie pretorsion z is a torsion if and only if $z(R) = 0$ ([2], prop. I.10.2).

08. A ring R is called

– *strongly prime* (SP), if $r(R) = 0$ for any proper pretorsion r of $R - Mod$ category;

– *left strongly semiprime* (SSP),if every essential ideal P is cofaithful, i.e. $(0 : P) = \bigcap_{\alpha=1}^{n} (0 : p_\alpha) = 0$ (essential ideal means a two-sided ideal that is essential as a left ideal).

Some descriptions of SSP-rings are obtained in the papers [4–7]. We present only a part of them.

The following conditions are equivalent:

(1) R is a SSP-ring.

(2) All pretorsions $r \geq z$ are torsions.

(3) Every proper pretorsion generates a proper torsion.

(4) R is a semiprime ring every nonzero ideal P of which possesses the property $(0 : P) = \left(0 : \hat{P}\right) = \bigcap_{\alpha=1}^{n} (0 : p_\alpha)$ for some elements $p_\alpha \in P$.

(5) The ring R is a finite subdirect sum of SP-rings.

09. Let $R = R_1 \oplus R_2 \oplus \ldots \oplus R_n$ be a ring direct sum. Denote by f_i the corresponding projections. There is a one-to-one correspondence between preradicals of $R - Mod$ and ordered n - tuples $(r_1,\ r_2, \ldots,\ r_n)$, where r_i is a preradical for R_i– *Mod*, given by $r \to (f_1[r], \ldots, f_n[r])$ and $(r_1,\ r_2, \ldots,\ r_n) \to \sum \{r_i\} f_i = \cap [r_i] f_i$. This correspondence preserves the elementary properties, intersections, sums, inclusions in both directions ([2], prop. I.9.1).

Now we begin the investigation of rings over which any preradical $r \geq r_R \quad (r \geq z)$ is a radical (torsion).

Proposition 1. *Every SSP-ring is a finite direct sum of indecomposable SSP-rings.*

Proof. We show that a SSP-ring does not contain any infinite direct sums of two-sided ideals ([4], prop. 6). Indeed, let us consider a direct sum $P = \sum_i \oplus P_i$ of ideals of the ring R. Then since R is semiprime we have $P \oplus (0 : P) \subseteq' R$. By assumption, R is a SSP-ring. Then the ideal $P \oplus (0 : P)$ is cofaithful and therefore $(0 : [P \oplus (0 : P)]) = \bigcap_{i=1}^{n} [0 : (p_i + p_i^*)]$ for some $p_i \in P$ and $p_i^* \in (0 : P)$. We show that in this case $P_\alpha = 0$ for any $\alpha = n+1,\ n+2,\ \ldots$. Indeed, from the equality $P_\alpha \cdot P_i = 0$ for any $\alpha \neq i$ we obtain $P_\alpha \cdot p_i = 0$. Besides that, the inclusion $P_\alpha \subseteq P \Rightarrow (0 : P) \subseteq (0 : P_\alpha) \Rightarrow p_i^* \in (0 : P_\alpha) \Rightarrow p_i^*\ P_\alpha = 0$ holds for any $i = \overline{1,\, n}$. Then $P_\alpha p_i^* \cdot\ P_\alpha p_i^* = 0$, therefore since R is semiprime we have $P_\alpha \cdot p_i^* = 0$. But then $P_\alpha (p_i + p_i^*) = 0 \Rightarrow P_\alpha \subseteq (0 : (p_i + p_i^*))$ for any $i = \overline{1,\, n}$. This means that $P_\alpha \subseteq \bigcap_{i=1}^{n} (0 : (p_i + p_i^*)) = 0$ and consequentely the considered direct sum is finite: $P = \sum_{i=1}^{n} \oplus P_i$. From this it follows that R does not contain any infinite sets of central and orthogonal indempotents because otherwise it would contain infinite direct sums of ideals. But the latter is equivalent to the decomposability of the ring R in a finite direct sum of SSP-rings. □

A pretorsion r of the category $R - Mod$ is called *superhereditary* (*stable*) if the class of r – torsion modules is closed with respect to direct products (essential extensions).

Superhereditarity of the pretorsion r is equivalent to the condition that its filter contains the least ideal P. It is denoted by $r_{(P)}$ and it is easy to verify that $r_{(P)}(M) = M \Leftrightarrow PM = 0$.

Lemma 2. *The following conditions are equivalent:*

(1) *All superhereditary pretorsions of $R - Mod$ are stable.*

(2) *All left ideals of the ring R are idempotent.*

(3) $(0 : M) = \left(0 : \hat{M}\right)$ *for any module $M \in R$ - Mod.*

Proof. Equivalence of (1) and (2) is proved in [2].

(1)⇒(3). Let M be an arbitrary module M for which $(0 : M) \neq 0$ because otherwise the equality $(0 : M) = \left(0 : \hat{M}\right)$ is obvions. Then the superhereditary pretorsion $r_{((0:M))}$ is stable. Since $r_{((0:M))}(M) = M$, we obtain $r_{((0:M))}\left(\hat{M}\right) = \hat{M}$, i.e. $(0 : M) \cdot \hat{M} = 0$ therefore $(0 : M) \subseteq \left(0 : \hat{M}\right)$. Now from the inclusion $\left(0 : \hat{M}\right) \subseteq (0 : M)$ we obtain $(0 : M) = \left(0 : \hat{M}\right)$.

(3)⇒(1). Let $r_{(P)}$ be an arbitrary superhereditary pretorsion. If $r_{(P)}(M) = M$

then $PM = 0$, therefore $P \subseteq (0 : M) = \left(0 : \hat{M}\right)$. It means $P\hat{M} = 0$ that is equivalent to the equality $r_{(P)}\left(\hat{M}\right) = \hat{M}$. Therefore $r_{(P)}$ is a stable pretorsion. □

Corollary 3. *Every indecomposable SSP-ring over which all left ideals are idempotent is simple.*

Indeed, let us consider an arbitrary essential ideal P of ring R and the torsion $\tau = r_{\widehat{R/P}}$. Then $\tau(R) = \left(0 : \widehat{R/P}\right) = (0 : R/P) = P$ (Lemma 2). Since $z(R) = 0$ it follows that $z < z \vee \tau$ and therefore $\tau(R) \subseteq (z \vee \tau)(R) \subseteq' R$. From the stability of the torsion $z \vee \tau$ (Statement 8) we obtain $(z \vee \tau)\left(\hat{R}\right) = (z + \tau)\left(\hat{R}\right) = z\left(\hat{R}\right) + \tau\left(\hat{R}\right) = \tau\left(\hat{R}\right) = \hat{R}$, therefore $\tau(R) = R$, i.e. $\tau = \varepsilon$. But then from the relations $\tau(R) = P = R$ it follows that the ring R does not contain any proper essential ideal. Let us now show that R is a simple rings. If K is a nonzero ideal of the ring R, then from its semiprimeness (R is an SSP-ring) it follows that the ideal $K \oplus (0 : K) \subseteq' R$. According to those proved earlier the ring $R = K \oplus (0 : K)$ and its indecomposability implies that $K = R$. In this way R is a simple ring. □

Corollary 4. *Any SSP-ring left ideals of which are indepotent is a finite direct sum of simple rings.*

Indeed, if R is a finite direct sum of rings R_α, then R is a SSP-ring left ideals of which are idempotent if in only if each direct summand R_α satisfies the same property. It remains us to use Proposition 1 and Corollary 3. □

Corollary 5. *If all preradicals of $R - Mod$ category are torsions then the ring R is a finite direct sum of simple rings with the same property.*

It is sufficient to show that the ring R satisfies the conditions of the previous Corollary 4.

Let us remark that R is a SSP-ring (Statement 08). Besides that, from the equality $r_M = r_{\hat{M}}$ it follows that any simple module is injective. Consequently, R is a left V-ring and therefore all its left ideals are idempotent ([2], prop. I.11.7). □

Corollary 6. *(Faith theorem). Any semiprime Goldie left V-ring is simple.*

This result follows directly from Corollary 3.

Corollary 7. *Any Goldie left V-ring is a finite direct sum of simple rings.*

It obviously follows from Corollary 4.

Lemma 8. *If all preradicals $r \geq r_R$ over ring R are radicals, then R is left strongly semiprime.*

Proof. We prove that any proper pretorsion r generates a proper torsion $\bar{r}$. Indeed, if $r \neq \varepsilon$ and $\bar{r} = \varepsilon$ then $r(R) \subseteq' R$. Consider the preradical $t = z + r_R + r$. Obviously, $t > r_R$ and $t > z$. By hypothesis, the preradical t is a radical and therefore $t(R/t(R)) = t(R/(z + r)(R)) = 0$. On the other hand, since the preradical $t > z$ and $t(R) = (z + r)(R) \subseteq' R$ we have $t(R/t(R)) = t(R/(z + r)(R)) =$

$R/(z+r)(R)$. For the equality $t(R/t(R)) = R/t(R) = 0$ it follows that $R = t(R) = (z+r)(R) = z(R) + r(R)$. Then $1 = x + y$ where $x \in z(R)$ and $y \in r(R)$. But then $(0:x) \cap (0\ :y) \subseteq (0:(x+y)) = 0$. From this and from $(0:x) \subseteq' R$ we have $(0\ :y) = 0 \in \mathrm{F}\ (r)$ where $\mathrm{F}\ (r)$ is the filter of pretorsion r. Consequently $r = \varepsilon$. The obtained contradiction shows that R is a SSP-ring. □

Lemma 9. *The following rings are simple:*

(1) *Indecomposable ring R over which all preradicals $r \geq r_R$ are radicals.*

(2) *Indecomposable self - injective SSP - rings.*

Prof. (1) Let P be a nonzero ideal of an indecomposable ring R over which all preradicals $r \geq r_R$ are radicals. By Lemma 8, R is a semiprime ring and therefore $P \oplus (0:P) \subseteq' R$. Consider the preradical $\tau = r_{R/P} + r_P$. Then $z = r_{\hat{R}} < r_R \leq r_P \leq \tau$ and $\tau(R) = \left(r_{R/P} + r_P\right)(R) \quad = r_{R/P}(R) + r_P(R) = (0:R/P) + (0:P) = P \oplus (0:P) \subseteq' R$. Since τ is radical $(\tau > r_R)$ we have $\tau(R/\tau(R)) = 0$. On the other hand, the relation $z \leq \tau$ and the inclusion $\tau(R) \subseteq' R$ imply $\tau(R/\tau(R)) = R/\tau(R)$. Then, from the last two equalities $\tau(R/\tau(R)) = 0 = R/\tau(R)$ we obtain that $R = \tau(R) = P \oplus (0:P)$, therefore, $P = R$ (the ring R is indecomposable). Consequently, R is a simple ring.

(2) Repeating proof of item (1) we have $\tau(R) = P \oplus (0:P) \subseteq' R$. According to the construction, we have $\tau \geq z$. Then, from the hypothesis (R is SSP-ring) and Statement 08, it follows that $h(\tau)$ is a stable torsion and therefore $h(\tau)\ (R) = R$. Self-injectivity of the ring R implies $h(\tau)\ (R) = \tau(R) = R = P \oplus (0:P)$, but its indecomposability implies that $P = R$. In this way, R is a simple ring. □

Theorem 10. *For self-injective ringR the following statements are equivalent:*

(1) *All preradicals $r \geq z$ are torsions.*

(2) *All preradicals $r \geq z$ are radicals.*

(3) *All pretorsions $r \geq z$ are torsions.*

(4) *The ring R is a finite direct sum of simple rings.*

(5) *The ring R is a finite direct sum of SP-rings.*

Proof. Implications (1)⇒(2)⇒(3) and (4)⇒(5) are obvious.

(3)⇒(4). According to Statement 08, R is a SSP-ring and, from Proposition 1, $R = \sum\limits_{\alpha=1}^{n} \oplus R_\alpha$, where R_α are indecomposable SSP-rings. Moreover, the rings R_α are self-injective because R itself is self-injective. Then by Lemma 9 item (2) R_α are simple rings, $\alpha = \overline{1,n}$.

(5)⇒(1). By the hypothesis $R = \sum\limits_{\alpha=1}^{n} \oplus R_\alpha$where R_α are self-injective SP - rings. Let K be one of these rings R_α. Consider an arbitrary proper preradical r of the category $R - Mod$ with the property $r \geq z$. Since K is a self-injective SP-ring, we have $r(K) = h(r)(K) = 0$ and, therefore $r \leq h(r) \leq r_K = z \leq r$, i.e. $r = z$ is a torsion. In this way over any direct summand R_α of the ring R every preradical $r \geq z$ is a torsion. Then R itself satisfies this property (Statement 09). □

Theorem 11. *The following conditions are equivalent:*
(1) *All preradicals* $r \geq r_R$ *of* $R - Mod$ *are radicals.*
(2) *The ring* R *is a finite direct sum of simple rings.*

Proof. (1)⇒(2). By Lemma 8, the ring R is a SSP-ring, and in according to Proposition 1 $R = \sum\limits_{\alpha=1}^{n} \oplus R_\alpha$, where R_α are indecomposable SSP-rings for any $\alpha = \overline{1, n}$. Besides that, by hypothesis and Statement 09, over each direct summand R_α all preradicals $r \geq r_{R_\alpha}$ are radicals. Then, according to Lemma 9 item(1), R_α are simple rings.

The implication (2)⇒(1) follows from Statement 09, because over any simple ring R all preradicals $r \geq r_R$ are radicals. □

Theorem 12. *The following conditions are equivalent:*
(1) *All preradicals* $r \geq z$ *of* $R - Mod$ *are torsions.*
(2) *The ring* R *is a finite direct sum of pseudoinjective simple rings.*

Proof. (1)⇒(2). By assumption and by Theorem 11, the ring R is a finite sum of simple rings. Let us show that each direct summand K of the ring R is a pseudoinjective ring i.e. we prove that $r_K = r_{\hat{K}}$. Indeed, since z is a torsions we have $z = r_{\hat{K}} \leq r_K$. From hypothesis, the radical r_K is also a torsion. Then, according to the Statement 06, $r_K = r_{\hat{K}}$ and therefore, K is a pseudoinjective ring.

(2)⇒(1). Let r be an arbitrary preradical of the pseudoinjective simple ring K. Then $r_K = r_{\hat{K}}$ (Statement 06) and every preradical r on the category $K - Mod$ with property $r \geq z$ is a torsion because the equality $r(K) = 0$ implies $r \leq r_K = r_{\hat{K}} = z \leq r$, therefore $r = z$. But then, by Statement 09 overing R, all preradicals $r \geq z$ also are torsions. □

References

[1] Kashu A.I. *Radicals and torsions in modules.* Kishinev, Ştiinţa, 1983 (In Russian).

[2] Bican L., Kepka T., Nemec P. *Rings, modules and preradicals.* Marcel Dekker, 1982.

[3] Kashu A.I. *When the radical associated to a module is a torsion*? Mat. Zametki, 1974, **16**, p. 41–48 (In Russian).

[4] Handelman D.E. *Strongly semiprime rings.* Pacif. J. Math., 1975, **60(1)**, p. 115–122.

[5] Kutami M., Oshiro K. *Strongly semiprime rings and nonsingular quasi-injective modules.* Osaka J. Math., 1980, **17**, p. 41–50.

[6] Bunu I.D. *On the strongly semiprime rings.* Bulet. A.Ş. R.M., Matematica, 1997, no. 1(23), p. 78–83.

[7] Van den Berg J.E. *Primeness Described in the Language of Torsion Preradicals.* Semigroup Forum, 2002, **64**, p. 425–442.

ASEM
str. Bănulescu Bodoni, 59
Chişinău, MD-2005 Moldova
E-mail: *ibunu@yahoo.com*

Received January 30, 2004