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Generating properties of biparabolic invertible polynomial maps in three variables

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Abstract. Invertible polynomial map of the standard 1-parabolic form $x_i \to f_i(x_1,\dots,x_{n-1}),\ \ i < n, x_n \to \alpha x_n + h_n(x_1,\dots,x_{n-1})$ is a natural generalization of a triangular map. To generalize the previous results about triangular and bitriangular maps, it is shown that the group of tame polynomial transformations TGA_3 is generated by an affine group AGL_3 and any nonlinear biparabolic map of the form $U_0 \cdot q_1 \cdot U_1 \cdot q_2 \cdot U_2$, where U_i are linear maps and both q_i have the standard 1-parabolic form.

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All invertible polynomial maps of the affine space A_n over a field K form the group GA_n (the affine Cremona group). It represents an important example of so called $Ind-$groups or $\infty-$dimensional algebraic groups (an inductive limit of finite dimensional algebraic varieties, see [1]).The elements of GA_n can be represented as tuples of polynomials

$$g = < f_1(x_1,\dots,x_n), f_2(x_1,\dots,x_n), ..., f_n(x_1,\dots,x_n) >, \tag{1}$$

which action on the volume form $dx_1 \wedge \dots \wedge dx_n$ is a multiplication it by a constant. It leads to the Jacobian condition

$$\det\left(\frac{\partial f_i}{\partial x_j}\right) = const; \tag{2}$$

$const \neq 0$. Remember that $Lie(GA_n) = ga_n$ consists of linear differential operators of the form

$$\sum_{i=1}^{n} a_i(x_1,\dots,x_n)\frac{\partial}{\partial x_i}, \tag{3}$$

where a_i are polynomials under the condition $\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} = const \in K$ It is well known (see [2]) that ga_n is a graded irreducible transitive algebra of a polynomial growth: $ga_n = \oplus_{k=-1}^{\infty} ga_n^{(k)}$, where homogeneous components $ga_n^{(k)}$ consist of the operators (3) for which $\deg a_i = k+1$.

There are important subgroups of GA_n:

(i) the affine group $AGL_n = GL_n \ltimes A_n^+ :\quad \deg f_i = 1, i = 1,2,\dots,n;$

(ii) B_n is a subgroup of triangular maps which elements have the form (1), where $f_i = f_i(x_1, \dots, x_i), i = 1, \dots, n$;

(iii) $GA_n^{(0)}$ is a stabilizer of zero and has a chain of normal subgroups $GA_n^{(0)} \triangleright GA_n^{(1)} \triangleright GA_n^{(2)} \triangleright \dots \triangleright GA_n^{(k)} \triangleright \dots$, which members $GA_n^{(k)}$ consist of the maps (1) of the type $f_i = x_i + \phi_i(x_1, \dots, x_n) + \dots$, where ϕ_i- are homogeneous $k+1-$ forms and $\dots$ means items of higher degrees, by the way, $GA_n^{(0)} = GL_n(K) \ltimes GA_n^{(1)}$;

(iv) the subgroup of tame maps TGA_n which are generated by the elementary transformations: $f_i = x_i,\ \ i \neq j, \quad f_j = x_j + h_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ and AGL_n.

As was shown in [3], $Lie(AGL_n)$ is a maximal subalgebra of the ga_n. The direct application of Shafarevich's theorem (see [1]) about the connection between Lie algebras and correspondent $\infty-$ dimensional algebraic groups leads to the conclusion: AGL_n is a maximal closed subgroup of GA_n. The subgroup B_n (Jonquièar's group) is a maximal solvable subgroup of GA_n and so can be considered as an analog of a Borel's subgroup. Remark that tuples of the form (1), which coordinates are formal power series without constant terms form a group with the composition of tuples as a group operation. It contains GA_n^0 as a subgroup. Moreover, the factors $GA_n^{(0)}/GA_n^{(k)}$ are finite dimensional algebraic groups.

Tame maps give most simple examples of nonlinear invertible polynomial maps. It is easy to see that $TGA_n =< AGL_n, B_n >$. As is well known, GA_2 has the structure of the amalgamated product: $GA_2 = AGL_2 * B_2$ and so $GA_2 = TGA_2$. In the dimension $n = 3$, I. Shestakov and U. Umurbaev in [4] have proved that Nagata's automorphism is wild, so TGA_3 is a proper subgroup of GA_3. Remark that if this automorphism is extended in a natural way to an automorphism of A_n for some $n > 3$ then this extension will be tame. As was mentioned above AGL_n is a maximal closed subgroup of GA_n. On the other hand, as follows from [5], a finite affine group nearly always is a maximal subgroup in the correspondent symmetrical group. So it is natural to investigate intermediate subgroups from the interval $AGL_n < TGA_n$. By using an amalgamated structure of GA_2 it isn't hard to construct such subgroups in the dimension $n = 2$. For example the groups $Q_m =< AGL_2, \sigma^{(m)} >$, where $\sigma^{(m)} =< x_1, x_2 + x_1^{m+1} >\in GA_2^{(m)} \cap B_2$ form an ascending chain $AGL_2 = Q_0 < Q_1 < \dots Q_m < Q_{m+1}, \dots$ and $GA_2 = \cup_m Q_m$. From the uniqueness of element's decomposition in amalgamated products it follows that all maps $\sigma^{(k)}, k > m$ don't belong to Q_m. As is well known, GA_3 has not such structure and to point out an intermediate subgroup isn't a simple task. It is easy to see that TGA_n can be defined also in such a manner $TGA_n =< B_n, AGL_n >$. In fact more strong result holds

Theorem 1. *([6]) Let t be an arbitrary nonlinear triangular map from B_n then*

$$TGA_n =< t, AGL_n > .$$

This theorem may be generalized so called standard 1-parabolic transformations.

Definition 1. *The transformation q of the form (1) is called standard 1-parabolic if there is an affine map A such that*

$$q^A =< f_1(x_1,\dots,x_{n-1}),\dots,f_{n-1}(x_1,\dots,x_{n-1}), x_n + f_n(x_1,\dots,x_{n-1}) > . \qquad (4)$$

Theorem 2. *Let q be an arbitrary nonlinear standard 1-parabolic transformation then*

$$TGA_n =< q, AGL_n > .$$

Proof. The result is a direct corollary of Theorem 1. Really, without lost of generality one can suppose that q has the form (4). Let $q^{-1} =< g_1,\dots,g_{n-1}, x_n - h_n(x_1,\dots,x_{n-1}) >$, then

$$g_i(f_1,\dots,f_{n-1}) \equiv x_i, f_n + h_n(f_1,\dots,f_{n-1}) \equiv 0. \qquad (5)$$

If all g_i are linear then the map q has the form $U \cdot t, U \in AGL_n, t \in B_n$. Otherwise, for number i such that g_i is nonlinear polynomial let us use the transvection $A_{n,i} = < x_1,\dots,x_{n-1}, x_n+x_i >\in AGL_n$ and get the element $q^{A_{n,i}} \cdot q^{-1} =< x_1,\dots,x_{n-1}, x_n - x_i + g_i >$ which is nonlinear triangular. □

Definition 2. *A map $q \in GA_n$ is called biparabolic if it can be represented as a composition of two standard 1-parabolic maps.*

In particular bitriangular maps, which were defined in [6] as maps of the kind $C_0 \cdot t_1 \cdot C_1 \cdot t_2 \cdot C_2,\ \ t_1, t_2 \in B_n, C_k \in AGL_n$, form a subclass of biparabolic ones. Let $G =< AGL_n, q >$, where q is a biparabolic map. Without lost of generality one can suppose that $q \in G$ has the form $q = q_1 \cdot q_2^A,\ A \in GL_n$. It is clear that standard 1- parabolic maps are permutable with the translations along the last coordinate $c_n : x_i \to x_i, i < n, x_n \to x_n + 1, 1 \in K$. This fact could be used for proving the same result ($G = TGA_n$) for biparabolic maps q. Really, the map q_2^A is permutable with the translation $c = c_n^A \in A_n^+$, so we can get the standard 1-parabolic map $q^c \cdot q^{-1} = q_1^c q_1^{-1} \in G$. Thus for most biparabolic maps the result can be deduced from Theorem 2. But it may happen that q_3 will be a linear map and the application of this theorem is impossible. In [7] (theorem 3) this situation was considered for bitriangular maps in the dimension $n = 3$. Next theorem is a generalization of that result.

Theorem 3. *Let q be an arbitrary nonlinear biparabolic transformation then*

$$TGA_3 =< q, AGL_3 > .$$

Proof. Let $G =< q, AGL_3 >$ As was mentioned above, we can suppose that $q = p_1 \cdot p_2^A$, where $p_1, p_2 \in GA_n^{(1)}$ (without linear parts). If $A = B_1 \cdot W \cdot B_2$ is a Brua decomposition, where W is a permutation matrix and B_1, B_2 are lower triangular matrices then we have $B_2 q B_2^{-1} = p_1^{B_2^{-1}} \cdot (p_2)^{B_1 \cdot W} \in G$. Since the maps $p_1^{B_2^{-1}}, (p_2)^{B_1}$

are standard 1-parabolic transformations also without linear parts, then without lost of generality one can suppose that $q = p_1 \cdot p_2^W$. Moreover, the maps $p_i^{(1,2)},\ \ i = 1, 2,$ are standard 1- parabolic ones and so one can suppose that there is an element $q \in G$ of the form

$$q = p \cdot p_1^{(1,3)}. \tag{6}$$

Let $p =< f_1(x_1, x_2), f_2(x_1, x_2), x_3 + f_3(x_1, x_2) >$ and $p^{-1} =< g_1(x_1, x_2), g_2(x_1, x_2), x_3 + g_3(x_1, x_2) >$ and identities (5) hold. If $\deg_{x_1} f_2 < \deg_{x_1} f_1$ then one can remove the map q by $(1, 2) \cdot q$, where $(1, 2) =< x_2, x_1, x_3 >$ is a transposition. So we can suppose that $\deg_{x_1} f_2 \geq \deg_{x_1} f_1$. On the other hand, if p has a decomposition $p = p'g$, where $g =< x_1 + h(x_2), x_2, x_3 >$, and p' has the form (4) then we can rewrite the map q in such a manner $q = p' \cdot (g^{(1,3)} \cdot p_1)^{(1,3)}$. Since $g^{(1,3)}$ is a triangular map then $g^{(1,3)} \cdot p_1$ is a 1-parabolic map. Hence, we can suppose also that p doesn't admit such decomposition $p = p' \cdot g$, where $h \not\equiv 0$.

Since the second factor of (6) is permutable with translation $c_1 =< x_1 + 1, x_2, x_3 >$, one can get an element $q_3 = q^{c_1} \cdot q^{-1} = p_1^{c_1} \cdot p_1^{-1} \in G$. As was mentioned above, the map q_3 has the form (4) and if it isn't a linear one then the result follows from Theorem 2. Let us investigate the situation when $q_3 = \Lambda \cdot x + z \in AGL_n$, here $\Lambda = (\lambda_{i,j}), i, j = 1, 2, 3, z = (z_1, z_2, z_3)$. The equality $p_1^{c_1} \cdot p_1^{-1} = q_3$ leads to the coordinate equalities

$$f_i(g_1 + 1, g_2) = \lambda_{i1}x_1 + \lambda_{i2}x_2 + \lambda_{i3}x_3 + z_i, i = 1, 2;$$
$$x_3 + g_3(x_1, x_2) + f_3(g_1 + 1, g_2) = \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3 + z_3.$$

By comparing the coefficients of x_3 we can obtain $\lambda_{i3} = 0, \lambda_{33} = 1$. If we take in account the identities (5) and act on the previous equalities by p we get

$$f_1(x_1 + 1, x_2) = \lambda_{11}f_1 + \lambda_{12}f_2 + z_1; \tag{7}$$
$$f_2(x_1 + 1, x_2) = \lambda_{21}f_1 + \lambda_{22}f_2 + z_2; \tag{8}$$

$$f_3(x_1 + 1, x_2) = \lambda_{31}f_1 + \lambda_{32}f_2 + f_3 + z_3.$$

Let us represent the coordinates of p in the form

$$f_i = \sum_{s=0}^{M_i} \phi_s^i(x_2)x_1^s,$$

$\phi_{M_i}^i \not\equiv 0,\ i = 1, 2, 3$. If $M_1 = M_2 = M$ then $M > 0$ and by comparing the coefficients of x_1^M in (7),(8) one can get

$$\phi_M^i = \lambda_{i1}\phi_M^1 + \lambda_{i2}\phi_M^2, \quad i = 1, 2.$$

If ϕ_M^1, ϕ_M^2 are linear independent polynomials over K then $\lambda_{i,j} = \delta_{i,j}$ (Kroneker's symbol). Comparing of the coefficients of x_1^{M-1} leads to the equality $M\phi_M^i + \phi_{M-1}^i = \phi_{M-1}^i$ which implies the contradiction $\phi_M^i \equiv 0, i = 1, 2$. So $\phi_M^2(x_2) = \mu\phi_M^1(x_2)$ for some $\mu \in K$. Let us use the transvection $U =< x_1 - \mu x_2, x_2, x_3 >$ and replace

$q \to U \cdot q$. In such a manner we get a map of the form (6) with $\phi^1_{M_1} \equiv 0$ in p i.e. for this map $M = M_2 > M_1$. Comparing the coefficients of x_1^M in identities (7),(8) leads to the equalities $0 = \lambda_{12}\phi^2_M, \phi^2_M = \lambda_{22}\phi^2_M$ which imply that $\lambda_{12} = 0, \lambda_{22} = 1$. Let us compare the coefficients of $x_1^{M_1-1}$:

$$\phi^1_{M-1} = \lambda_{11}\phi^1_{M-1}, \quad M\phi^2_M + \phi^2_{M-1} = \lambda_{21}\phi^1_{M-1} + \phi^2_{M-1}.$$

It follows that $\phi^1_{M-1} \not\equiv 0$ $(M_1 = M-1)$ and $\lambda_{11} = 1$. It is clear that the highest degree of x_1 which can be present by jacobian of the pair(f_1, f_2) does not exceed $2M-2$. With regard to the equality $M\phi^2_M = \lambda_{21}\phi^1_{M-1}$, the jacobian condition (2) leads to the identity $\phi^2_M \cdot \frac{d\phi^2_M}{dx_2} = 0$, hence, $\phi^2_M = const$. If $M > 2$ then comparing the coefficients of x_1^{M-2} in (7) leads to the contradiction $(M-1)\phi^1_{M-1} + \phi^1_{M-2} = \phi^1_{M-2}$, i.e.$\phi^1_{M-1} = 0$. Hence, $M = 2$ or $M = 1$. In the first case from (7) we have $\phi^1_1 = z_1$. The equalization of monomials without x_1 in (8) leads to the equality

$$\frac{M(M-1)}{2}\phi^2_2 + (M-1)\phi^2_1 + \phi^2_0 = \lambda_{21}\phi^1_0 + \phi^2_0 + z_2,$$

which under $M = 2$ implies $\phi^2_1 = \mu\phi^1_0 + const, \mu \in K$. After all we obtain that

$$f_1 = z_1x_1 + \phi^1_0(x_2), \quad f_2 = \phi^2_2 x_1^2 + \left(\mu\phi^1_0(x_2) + const\right) x_1 + \phi^2_0(x_2).$$

This implies that p can be decomposed in such a manner

$$p =< z_1x_1, \phi^2_2(x_1 - (z_1)^{-1}\phi^1_0(x_2))^2 + \left(\mu\phi^1_0(x_2) + const\right)(x_1 - (z_1)^{-1}\phi^1_0(x_2)) + \phi^2_0(x_2),$$
$$x_3 + f_3(x_1 - (z_1)^{-1}\phi^1_0(x_2), x_2) > \cdot < x_1 + (z_1)^{-1}\phi^1_0(x_2), x_2, x_3 > .$$

But, as was mentioned above, the map p doesn't admit such decomposition and so $\phi^1_0(x_2) \equiv 0$. Thus $p =< z_1x_1, \phi^2_2x_1^2 + const x_1 + \phi^2_0(x_2), x_3 + f_3(x_1, x_2) >$ is a triangular map. In the case $M = 1$ it is evident that the map $(1,2) \cdot t$ is a triangular one. On the other hand, we can repeat our reasoning for the map $\hat{q} = q^{-(1,3)} = p_1^{-1} \cdot p^{-(1,3)}$ and conclude that p_1 is also triangular. This means that in fact, the situation when $q_3 = q^{c_1} \cdot q^{-1}, q_4 = \hat{q}^{c_1} \cdot \hat{q}^{-1} \in AGL_n$, can be realized when both elements owe triangular ones i.e. when q is bitriangular. So the result follows from Theorem 3 from [7]. □

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