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Special radicals of graded rings

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Abstract. We consider the graded radicals of graded rings, and prove that any radical in the category ring graded by a group G can be defined by means of some class of graded modules. We also describe the classes of graded modules for special graded radical.

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1 Introduction

The general theory of radicals of rings and algebras began to develop in papers of A.G. Kurosh and S.A. Amitsur in the 1950s. They observed that the general theory of radicals can be developed in any algebraic systems which the concept of the kernel homomorphism with usual properties takes place in, i.e. in the "good" categories.

For basic notions and terminology on general theory of radicals we refer the reader to the monograph [1].

For given radical $\mathcal{R}$ in the category of associative rings and ring homomorphisms there are different ways of defining a graded version of $\mathcal{R}$:

At first, one may consider a natural definition for graded version of $\mathcal{R}_G$ in category of graded rings and graded-preserving ring homomorphisms.

Secondly, for defining a graded version of $\mathcal{R}$ for a graded ring A one can consider $\mathcal{R}(A)_G$, the largest graded ideal contained in $\mathcal{R}(A)$.

At last, it is possible to consider the largest graded ideal I of A such that $I \bigcap A_e = \mathcal{R}(A_e)$, where A_e is an identity graded component of A.

Besides using the generalized smash product, M. Beattie and P. Stewart [2] introduced a method for defining a so called reflexive radical $\mathcal{R}_{ref}$. They investigated the properties of reflexive radicals and compared them with graded radicals which had been previously studied.

The graded radicals of graded rings have been investigated in papers ([3],[4],[5],[6]).

On the other hand, in 1962 V.A. Andrunakievich and Yu.M. Ryabuhin showed that any special radical of an associative ring can be defined by means of some class of modules.

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The purpose of this paper is to prove analogous results for special radicals of category of the graded rings and to define the classes of graded modules corresponding to classical graded radicals.

2 Preliminaries

Let A be an associative ring (not necessaryly with identity), G a group with identity e.

A ring A is called *G-graded* (or simple *graded*) if there exists a family $\{A_g \mid g \in G\}$ of additive subgroups of A such that $A = \bigoplus_{g \in G} A_g$ and $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$.

If a ring A has an identity 1, then $1 \in A_e$.

The elements of the set $h(A) = \bigcup\limits_{g \in G} A_g$ are called *homogeneous elements* of the ring A. A nonzero element $r_g \in A_g$ is said to be *homogeneous of degree g*.

Any nonzero r has a unique expression as a sum of homogeneous elements, $r = \sum\limits_{g \in S} r_g$, where r_g is nonzero for a finite number of $g \in G$. The nonzero elements r_g in the decomposition of r are called *homogeneous components* of r.

An ideal I is called *graded* (or *homogeneous*) if $I = \bigoplus_{g \in G}(I \cap A_g)$. For any ideal I of A (left, right or two-sided) the largest graded ideal of A contained in I will be denoted by I_G.

Let $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ be G-graded rings. A ring homomorphism $f : A \longrightarrow B$ is called graded-preserving if $f(A_g) \subseteq B_g$ for all $g \in G$.

The category of G-graded rings $G\mathcal{R}ings$ consists of G-graded rings and graded-preserving homomorphisms.

Let A be a G-graded ring. A right A-module M is called a *G-graded A-module* if there exists a family $\{M_g \mid g \in G\}$ of additive subgroups of M such that $M = \bigoplus_{g \in G} M_g$ and $M_g A_h \subseteq M_{gh}$ for all $g, h \in G$.

Let N and M be G-graded right A-module. A homomorphism $f : N \to M$ is called a *graded morphism of degree h* if $f(N_g) \subseteq M_{hg}$ for any $g \in G$. All graded morphisms of degree h form the additive subgroup $HOM(N_R, M_R)_h$ of the group $Hom(N_R, M_R)$.

A submodule $N \subseteq M$ is called a *graded submodule* if $N = \bigoplus_{g \in G}(N \cap M_g)$. In other words, $N \subseteq M$ is a graded submodule if for any $x \in N$ it follows that N contains all homogeneous components of x.

Let M be a graded module, N its graded submodule. Then M/N may be made into a graded module by putting $(M/N)_g = (M_g + N)/N$ for all $g \in G$. With this definition, the canonical projection $M \to M/N$ is a graded morphism of degree e.

Further details on graded rings and modules may be found in [3]

3 Graded modules and radicals

Let A be a ring, M a right A-module. The set $Ann_A(M) = \{a \in A | Ma = 0\}$ is called an annihilator of a A-module M. Recall that a module M is called faithful if $Ann_A(M) = 0$.

In [8] V.A. Andrunakievich and Yu.M. Ryabuhin defined a general class Σ of modules in the following way. For every associative ring A they denote some class of nontrivial right A-modules (can be empty) by Σ_A . The set

$$Ker\Sigma_A = \bigcap\{Ann_A(M)|M \in \Sigma_A\}$$

is called the kernel of the class Σ_A. If $\Sigma_A = \emptyset$ then let‘s assume $Ker\Sigma_A = A$.

The class Σ of all Σ_A is called a *general class of the modules*, if the following hold:

P1. If $M \in \Sigma_{A/B}$, then $M \in \Sigma_A$.

P2. If $M \in \Sigma_A$ and $B \subseteq Ann_A(M)$, then $M \in \Sigma_{A/B}$.

P3. If $Ker\Sigma_A = 0$, then $\Sigma_B \neq \emptyset$ for any nonzero ideal B of A.

P4. If $\Sigma_B \neq \emptyset$ for any nonzero ideal B of A, then $Ker\Sigma_A = 0$.

Using the general class of modules, they defined Σ-radical

$$\mathcal{R}(\Sigma, A) = Ker\Sigma_A = \bigcap\{Ann_A(M)|M \in \Sigma_A\}$$

for any associative ring A and proved that Σ-radical is a Kurosh-Amitsur radical. Conversely, if $\mathcal{R}$ is a radical in the category of associative rings, then there is a general class of modules Σ such that $\mathcal{R}$ is equal to Σ-radical ([8], Theorem 1).

In this case any Σ-semisimple ring is the subdirect product of a family of rings from $\mathcal{L}(\Sigma)$, where $\mathcal{L}(\Sigma)$ is the class of all rings A that have a faithful A-module from Σ_A ([8], Corollary 4).

Since the category of G-graded rings $G\mathcal{R}ings$ has all necessary properties the general theory of radicals in sense of Kurosh-Amitsur is valid.

It is straightforward to check that if M is a graded A-module, then the annihilator $Ann_A(M)$ is the graded ideal in A. We have the following

Proposition 1. *Let A be a G-graded ring, B a graded ideal of A. If M is a graded A/B-module, then M becomes a graded A-module by setting $xa = x(a + B)$ and $B \subseteq Ann_A(M)$. Conversely, if M is a graded A-module and B is a graded ideal, such that $B \subseteq Ann_A(M)$, then M is a graded A/B-module by setting $x(a+B) = xa$. Any graded submodule of A/B-module M is a graded submodule of A-module M too, and conversely. Moreover, $Ann_{A/B}(M) = Ann_A(M)/B$ (as graded ideals).*

Proof. The proof follows from ([8], Proposition 1) as the given definition is coordinated with grading.

On the other hand, it is possible to define *the general class of G-graded modules* Σ_G as the class all Σ_{GA} satisfying conditions GP1-GP4, which are obtained from the corresponding conditions P1-P4 by replacement of the word "ideal" by the word "graded ideal" (here Σ_{GA} is some class of right graded modules over G-graded ring A.)

Then the *graded Σ_G-radical* of G-graded ring A is defined as

$$\mathcal{R}(\Sigma_G, A) = Ker\Sigma_{GA} = \bigcap\{Ann_A(M)|M \in \Sigma_{GA}\}$$

Theorem 2. *Let Σ_G be a general class of G-graded modules, then Σ_G-radical is a radical in the category of graded rings $G\mathcal{R}ings$. Conversely, if $\mathcal{R}$ is a radical in the category $G\mathcal{R}ings$, then there exists a general class of G-graded modules Σ_G such that $\mathcal{R}$ coincides with the Σ_G-radical.*

Moreover, any Σ_G-semisimple graded ring is the subdirect product of graded rings A which have faithful A-modules from the class Σ_{GA}.

Proof. The proof of this theorem using Proposition 1 and conditions GP1-GP4 is essentially the same as that in the ungraded case (see [8], Theorem 1).

4 Special graded radicals

Recall that a special radical $R_{\mathfrak{M}}(A)$ of an associative ring A is an upper radical, defined by any special class of rings $\mathfrak{M}$, i.e. for any ring A, $R_{\mathfrak{M}}(A)$ is equal to the intersection of all ideals P of A such that $A/P \in \mathfrak{M}$ ([1], chapter 3)

In [7] the general classes of modules, corresponding to special radicals, are called *the special classes of modules* and the special classes of modules for classical special radicals were determined.

Using the concept of a special radical in categories ([1], Chapter 5) we can define a special radical in the category of G-graded rings $G\mathcal{R}ings$.

Let $\mathfrak{M}$ be some class of graded rings, then a graded ideal P of a G-graded ring A is called an $\mathfrak{M}$-ideal if $A/P \in \mathfrak{M}$.

A class $\mathfrak{M}$ of G-graded rings is said to be *a special class* if it satisfies the following conditions:

M1. If B is a graded ideal of a ring A and P is an $\mathfrak{M}$-ideal in A which does not contain of B, then $P \bigcap B$ is a proper $\mathfrak{M}$-ideal in B.

M2. If Q is a proper $\mathfrak{M}$-ideal in B and B is a graded ideal of a ring A, then there exists only one $\mathfrak{M}$-ideal P in A such that $P \bigcap B = Q$.

Proposition 3. *Let $\mathfrak{M}$ be a special class of graded rings, Σ_G a general class of graded modules such that the special radical $R_{\mathfrak{M}}$ is equal to Σ_G-radical. Then a graded ideal P of a ring A is an $\mathfrak{M}$-ideal if and only if $P = Ann_A(M)$ for some graded A-module $M \in \Sigma_{GA}$.*

Proof. Let P be a graded $\mathfrak{M}$-ideal of a ring A. From Theorem 2 we have that there exists a faithful graded A/P-module $M \in \Sigma_{G(A/P)}$. By (GP1), M belongs to Σ_{GA}. Hence $P = Ann_A(M)$ by Proposition 1.

Conversely, if $P = Ann_A(M)$ for some graded A-module $M \in \Sigma_{GA}$, then by Proposition 1 and (GP2) we obtain that M is a faithful graded A/P-module. Therefore $A/P \in \mathfrak{M}$. The proof is complete.

From ([1], Chapter 5) we have that there is the largest special class of graded rings $\mathfrak{P}$.

Recall that a graded ideal P of a graded ring A is said to be *gr-prime* if for any graded ideals I, J such that $IJ \subseteq P$ we have either $I \subseteq P$ or $J \subseteq P$. A graded ring is called *gr-prime* if (0) is the gr-prime ideal.

Proposition 4. *The largest special class $\mathfrak{P}$ of graded rings coincides with the class of all gr-prime rings.*

Proof. Assume that $A \in \mathfrak{P}$ and A is not gr-prime. Then there are nonzero graded ideals I and J such that $IJ = 0$. From ([1], Ch.5, §5, Proposition5) we have that $K = I \bigcap J \neq 0$ is a graded ideal in A, $K^2 \subseteq IJ = 0$ and $K \in \mathfrak{P}$. Now consider $E = K \oplus K$. Then the ideals $P_1 = \{(k, 0)|k \in K\}$, $P_2 = \{(0, k)|k \in K\}$ and $B = \{(k, k)|k \in K\}$ belong to $\mathfrak{P}$, but for $\mathfrak{P}$-ideals P_1 and P_2 we have $P_1 \bigcap B = P_1 \bigcap B = 0$. This contradicts condition M2.

Let's show that the class of all gr-prime rings is special.

Let P be a gr-prime ideal of A, B a graded ideal of A, and $IJ \subseteq P \bigcap B$ for some graded ideals I and J in B. Denote by I_A and J_A the ideals in A, generated by I and J respectively. Since $I_A^3 \subseteq I \subseteq I_A$, $J_A^3 \subseteq J \subseteq J_A$, then $I_A^3 J_A^3 \subseteq P$. As an ideal P is gr-prime, then either $I_A^3 \subseteq P$ or $J_A^3 \subseteq P$. Hence either $I \subseteq P \bigcap B$ or $J \subseteq P \bigcap B$, therefore $Q = P \bigcap B$ is gr-prime. Thus condition M1 is carried out.

Let Q be a proper gr-prime ideal of B. Define the set

$$P_0 = \{a \in A | aB \subseteq Q, \quad Ba \subseteq Q\}.$$

As Q is gr-prime it is straightforward to check that P_0 is a gr-prime ideal in A and $P_0 \bigcap B = Q$. Let P be any gr-prime ideal in A such that $P \bigcap B = Q$. Since Q is gr-prime, $P \subseteq P_0$. On the other hand, $BP_0 \subseteq P_0 \bigcap B = Q \subseteq P$. Therefore by primeness of P we have $P_0 \subseteq P$. Thus condition M2 is carried out. The proof is complete.

From Proposition 4 and ([1], Ch. 5, §5, Theorem 4) we shall receive the following description of special classes of graded rings.

A class $\mathfrak{M}$ of graded rings is *special* if and only if the following hold:

GA1. All rings belonging to $\mathfrak{M}$ are gr-prime.

GA2. If $A \in \mathfrak{M}$ and I is a nonzero graded ideal of a ring A, then $I \in \mathfrak{M}$.

GA3. If B is a graded ideal of a gr-prime ring A and $B \in \mathfrak{M}$, then $A \in \mathfrak{M}$.

Consider now special graded radicals.

The graded Jacobson radical. Recall that a graded right A-module M is *gr-irreducible* if $MA = M$ and M does not contain non-trivial graded submodules. A graded ring A is *gr-primitive* (right) if there is a faithful gr-irreducible right A-module.

The graded Jacobson radical of A, $\mathcal{J}_G(A)$, may be defined equivalently :

(1) [3] the intersection of all annihilators of gr-irreducible right A-modules,

(2) [3] $\mathcal{J}_G(A)$ is also the intersection of the maximal graded right ideals of A,

(3) the special radical, defined by the class of all gr-primitive rings.

Thus, if Σ_G is the class of all gr-irreducible right A-modules, then $\mathcal{J}_G(A)$ is a Σ_G-radical.

In [9] G. Abrams and C. Menini defined for semigroup-graded rings A, the graded Jacobson radical $\mathcal{J}_{gr}(A)$ as the intersection of all annihilators of gr-irreducible $*$-graded A-modules. For a semigroup S (possibly with zero ν) a S-graded module M is called $*$-graded if its ν-component M_ν is equal to (0). They provided various conditions on A which imply that $\mathcal{J}_{gr}(A) \subseteq \mathcal{J}(A)$.

Remark 5. Note that definitions (1)–(3) above for graded Jacobson radical are not equivalent if we consider rings graded by arbitrary semigroups, as in this case the annihilator $Ann_A(M)$ of graded A-modules can be an ungraded ideal.

Example 6. Let S be the simplest rectangle band, i.e. a semigroup $S = \{(m,n) \mid m,n = 1,2\}$ with multiplication defined by $(x,y)(z,t) = (x,t)$. Consider a semigroup ring $A = kS$ with coefficients in a field k, S-graded in the usual way.

Let M be a gr-irreducible right A-module, then $M = mA$ for any nonzero homogeneous element m from M. Hence, the element $(1,1)-(2,1)$ belongs to $Ann_A(M)$ for any gr-irreducible A-module M, and thus $(1,1)-(1,2) \in J_{gr}(A)$.

On other hand, $(1,1) \notin Ann_A(P)$ for gr-irreducible A-module $P = (1,1)A$, and consequently $(1,1) \notin J_{gr}(A)$. Thus in this case $\mathcal{J}_{gr}(A)$ is an ungraded ideal of A.

The graded prime radical. The graded prime radical of A $\mathcal{B}_G(A)$ is the intersection of all gr-prime ideals of A ([3],[4]).

In [6] a graded right A-module M is called *gr-prime* if for every nonzero graded submodule N of M and every graded ideal I of A, $NI = 0$ implies $I \subseteq Ann_A(M)$. They defined the graded prime radical $\mathcal{B}_G(A)$ as the intersection of the annihilators of gr-prime modules.

It is straightforward to check that these definitions are equivalent.

Thus $\mathcal{B}_G(A)$ is the smallest graded special radical and it is the Σ_G-radical, generated by the class Σ_G of all gr-prime modules.

The graded Levitzki radical. For a graded ring A the graded Levitzki radical $\mathcal{L}_G(A)$ is the intersection of the gr-prime ideals P of A such that A/P has no nonzero graded locally nilpotent ideal [2].

Since $\mathcal{L}_G(A) = \mathcal{L}(A)_G$ ([2], proposition 3.2), $\mathcal{L}_G(A)$ is the largest locally nilpotent graded ideal.

A gr-prime A-module M is called a *graded Levitzki A-module* if $A/Ann_A(M)$ has no nonzero graded locally nilpotent ideal. Let Σ_G be the class of all graded Levitzki modules. Then the Σ_G-radical $R(\Sigma_G, A)$ coincides with the graded Levitzki radical $\mathcal{L}_G(A)$.

The graded Köthe radical. For a graded ring A the graded Köthe radical $\mathcal{K}_G(A)$ is the largest graded nilideal. It is clear that $\mathcal{K}_G(A) = (\mathcal{K}(A))_G$ and $\mathcal{K}_G(A)$ is the intersection of the gr-prime ideals P of A such that A/P has no nonzero graded nilideal.

A gr-prime A-module M is called a *graded Köthe A-module* if $A/Ann_A(M)$ has no nonzero graded nilideal. Then the graded Köthe radical $\mathcal{K}_G(A)$ is the Σ_G-radical, generated by the class of all graded Köthe modules.

The graded Brown-McCoy radical. For a graded ring A the graded Brown-McCoy radical $\mathcal{U}_G(A)$ is the intersection of the graded ideals P of A such that A/P is a graded simple ring with identity.

For every G-graded ring A, $\mathcal{U}(A)_G \subseteq \mathcal{U}_G(A)$, and this inclusion may be proper. ([2], proposition 3.5).

Like in [7], a graded right A-module M is called *gr-simple* if $MA \neq 0$ and for every graded ideal I of A such that $MI \neq 0$, there exits $b \in I_e$ such that $mb = m$ for all $m \in M$.

It is straightforward to check that the class Σ_G of all gr-simple modules defines the Σ_G-radical that coincides with the graded Brown-McCoy radical $\mathcal{U}_G(A)$.

The graded compressive radical. Recall that $\mathcal{A}(A)$, the compressive radical of a ring A, is the intersection of all ideals I of A such that A/I has no zero divisor.

It is straightforward to check that the class of all graded rings which have no homogeneous zero divisor is special.

For a graded ring A the graded compressive radical $\mathcal{A}_G(A)$ is the intersection of the graded ideals I of A such that A/I has no homogeneous zero divisor.

Recall that a nonzero element $m \in M_A$ is called a zero divisor if there exits $a \in A, \quad a \notin Ann_A M$ such that $ma = 0$. The class Σ_G of all graded modules which have no homogeneous zero divisor defines the graded compressive radical $\mathcal{A}_G(A)$.

Remark 7. Note that these results will be true if we consider rings graded by cancellative semigroups. However, for an arbitrary semigroup it does not hold.

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