BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(44), 2004, Pages 15–25
ISSN 1024–7696

Semitopological isomorphism of topological groups

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Abstract. A criterion of the continuous isomorphism of topological groups to be semitopological is obtained in the article.

Mathematics subject classification: 16W80.

Keywords and phrases: Topological group, normal subgroup, quotient group, topological homomorphism, continuous isomorphism..

The following isomorphism theorem is often used in algebra:

If R is a group (a ring), I its normal subgroup (ideal) and A is a subgroup (subring) in R, then the groups (rings) $A/(A \cap I)$ and $(A+I)/I$ are isomorphic.

The similar theorem is not valid for topological groups (topological rings), but the following one is:

If $(R,\ \tau)$ is a topological group (topological ring), [1] I is a normal subgroup (ideal) in R and A is a subgroup (subring) in R then the canonical isomorphism which maps the topological group (topological ring) $(A,\ \tau\mid_A)/(A \cap I)$ to the topological group (topological ring) $(A+I,\ \tau|_{A+I})/I$ is continuous.

It follows from Theorem 1 that the assertion on the continuity of the canonical isomorphism has no generalization.

The case when A is a normal subgroup in the group R, respectively, ideal in the ring R is often considered in the theory of group and the theory of rings, especially in the radical theory of groups and rings. The canonical isomorphism possesses additional properties in this case. The notion of the semitopological isomorphism of topological groups is introduced in the article for their study (see Definition 2).

The notion of semitopological isomorphism and the study of its properties for topological rings were given in [1].

The semitopological isomorphism can be considered not only in the class of all topological groups but also in its subclasses, (in particular, for the class of all Hausdorff topological groups and other classes).

Theorem 4 is a criterion for a continuous isomorphism to be semitopological and is the main result of the article. It is proved that the property of an isomorphism to be semitopological is kept by operations of taking subgroups (Theorem 7), quotient groups (Theorem 8) and direct products (Theorem 9).

1 Theorem. *If $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a continuous isomorphism of topological groups (topological rings) $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$, then there exists a topological group*

[1] A topological group (topological ring) is not supposed to be Hausdorff.

(topological ring) $(\widehat{G},\ \widehat{\tau})$ [2] *and a topological (i.e. open and continuous) homomorphism* $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$*, such that the following assertions hold:*

$\widehat{\tau}\mid_G = \tau$*, i.e.* $(G,\ \tau)$ *is a subgroup (subring) of the topological group (topological ring)* $(\widehat{G},\ \widehat{\tau})$*;*

$\widehat{\xi}\mid_G = \xi$*, i.e. the homomorphism* $\widehat{\xi}$ *is an extension of the isomorphism* ξ*.*

Proof. Consider a topological group (topological ring) $(\widehat{G},\ \widehat{\tau})$ which is equal to the direct product of topological groups (topological rings) $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$.

If $G' = \{\big(g,\ \xi(g)\big) \mid g \in G\}$, then G' is a subgroup (subring) of the group (ring) $\widehat{G}$.

Define a mapping $\xi' : G \to G'$ as follows: $\xi'(g) = (g,\ \xi(g))$.

Prove that $\xi' : (G,\ \tau) \to (G',\ \widehat{\tau}\mid_{G'})$ is a topological isomorphism of topological groups (topological rings).

Indeed, since $\xi : G \to \overline{G}$ is an isomorphism, then so is ξ'.

If U is an arbitrary neighbourhood of the identity (zero) in $(G',\ \widehat{\tau}\mid_{G'})$, then there exist neighbourhoods of the identities (zeroes) V and $\overline{V}$ in topological groups (topological rings) $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ respectively such that $\{(g,\ \overline{g}) \mid g \in V,\ \ \overline{g} \in \overline{V}\} \cap G' \subseteq U$. Since $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a continuous isomorphism then there exists a neighbourhood of the identity (zero) V_1 in $(G,\ \tau)$ such that $V_1 \subseteq V$ and $\xi(V_1) \subseteq \overline{V}$. Hence

$$\xi'(V_1) = \{\big(g,\ \xi(g)\big) \mid g \in V_1\} \subseteq \{(g,\ \overline{g}) \mid g \in V,\ \ \overline{g} \in \overline{V}\} \cap G' \subseteq U,$$

and therefore $\xi' : (G,\ \tau) \to (G',\ \widehat{\tau}_{G'})$ is a continuous isomorphism.

If now V is an arbitrary neighbourhood of the identity (zero) in $(G,\ \tau)$ then $W = \{(g,\ \overline{g}) \mid g \in V,\ \ \overline{g} \in \overline{G}\}$ is a neighbourhood of the identity (zero) in $(\widehat{G},\ \widehat{\tau})$, and hence $W \cap G'$ is a neighbourhood of the identity (zero) in $(G',\ \widehat{\tau}\mid_{G'})$. Since

$$\xi'(V) = \{\big(g,\ \xi(g)\big) \mid g \in V\} \supseteq W \cap G',$$

then $\xi'(V)$ is a neighbourhood of the identity (zero) in $(G',\ \widehat{\tau}\mid_{G'})$. Hence $\xi' : (G,\ \tau) \to (G',\ \widehat{\tau}_{G'})$ is proved to be an open isomorphism and therefore ξ' is a topological isomorphism.

When identifying the element $g \in G$ with the element $\big(g,\ \xi(g)\big) \in G'$ we obtain that the topological group (topological ring) $(G,\ \tau)$ is a subgroup (subring) of the topological group (topological ring) $(\widehat{G},\ \widehat{\tau})$ and $\xi(g) = \xi\big(g,\ \xi(g)\big)$.

It remains to prove that there exists a topological homomorphism $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ which is an extension of the isomorphism ξ.

Define a mapping $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ as follows: $\widehat{\xi}(g,\ \overline{g}) = \overline{g}$. Then it is a topological homomorphism.

Since $\widehat{\xi}(g,\ \xi(g)) = \xi(g)$ then $\widehat{\xi}$ is a topological homomorphism extending the isomorphism ξ, that completes the proof. □

[2] It is clear that if the topological group (topological ring) $(\overline{G},\ \overline{\tau})$ is Hausdorff then so is $(G,\ \tau)$. In this case without loss of generality $(\widehat{G},\ \widehat{\tau})$ is also assumed to be so, otherwise $(\widehat{G},\ \widehat{\tau})$ is replaced by $(\widehat{G},\ \widehat{\tau})/\widehat{I}$, where $\widehat{I} = [\{e\}]_{(\widehat{G},\ \widehat{\tau})}$.

2 Definition. *A continuous isomorphism* $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ *of topological groups is said to be a semitopological isomorphism if there exists a topological group* [3] $(\widehat{G},\ \widehat{\tau})$ *and a topological (i.e. open and continuous) homomorphism* $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ *such that the following assertions hold:*

G is a normal subgroup in the group $\widehat{G}$;

$\widehat{\tau}\ |_G = \tau$, i.e $(G,\ \tau)$ is a subgroup of the topological group $(\widehat{G},\ \widehat{\tau})$;

$\widehat{\xi}\ |_G = \xi$, i.e. the homomorphism $\widehat{\xi}$ is an extension of the isomorphism ξ.

3 Proposition. *Let $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ be topological groups and $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ be a semitopological isomorphism. If $(\widehat{G},\ \widehat{\tau})$ is a topological group and $\widehat{\xi}$ a topological homomorphism mentioned in Definition 2, then the following assertions hold:*

1) $G \cap \ker \widehat{\xi} = \{e\}$;

2) for every $\widehat{g} \in \widehat{G}$ there exist the only pair of elements $g \in G$ and $b \in \ker \widehat{\xi}$ such that $\widehat{g} = g \cdot b$;

3) $c \cdot h \cdot c^{-1} = h$ for every $c \in \ker \widehat{\xi}$ and $h \in G$.

Proof. Since ξ is an isomorphism and $\widehat{\xi}\ |_G = \xi$ then $\ker \widehat{\xi} \cap G = \ker \xi = \{e\}$, that proves the assertion 1.

2) Let $\widehat{g} \in \widehat{G}$ and $\overline{g} = \widehat{\xi}(\widehat{g}) \in \overline{G}$. Since $\xi : G \to \overline{G}$ is a bijection then there exists the only element $g \in G$ such that $\xi(g) = \overline{g}$. Consider the element $b = g^{-1} \cdot \widehat{g}$. Hence $g \cdot b = \widehat{g}$ and

$$\widehat{\xi}(b) = \widehat{\xi}(g^{-1} \cdot \widehat{g}) = \widehat{\xi}(g^{-1}) \cdot \widehat{\xi}(\widehat{g}) = \overline{g}^{-1} \cdot \overline{g} = \overline{e},$$

i.e. $b \in \ker \widehat{\xi}$. That completes the proof of the assertion 2.

3) Let $c \in \ker \widehat{\xi}$ and $h \in G$. Since G is a normal subgroup of $\widehat{G}$ then $c \cdot g \cdot c^{-1} \in G$. Hence

$$\xi(c \cdot h \cdot c^{-1}) = \widehat{\xi}(c \cdot h \cdot c^{-1}) = \widehat{\xi}(c) \cdot \widehat{\xi}(h) \cdot \widehat{\xi}(c^{-1}) = \overline{e} \cdot \widehat{\xi}(h) \cdot \overline{e} = \widehat{\xi}(h) = \xi(h).$$

Since ξ is an isomorphism then $c \cdot h \cdot c^{-1} = h$, that completes the proof of the proposition. □

4 Theorem. *If $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a continuous isomorphism of topological groups $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$, then the isomorphism ξ is semitopological iff the following two conditions hold:*

1. For every neighbourhood V_0 of the identity of the topological group $(G,\ \tau)$ there exist neighbourhoods $\overline{V}_1$ and V_1 of the identity in $(\overline{G},\ \overline{\tau})$ and $(G,\ \tau)$, respectively, such that $v \cdot V_1 \cdot v^{-1} \subseteq V_0$ for every $v \in \xi^{-1}(\overline{V}_1)$;

2. For every neighbourhood V_0 of the identity in the topological group $(G,\ \tau)$ and every element $g \in G$ there exists a neighbourhood $\overline{V}_1$ of the identity in $(\overline{G},\ \overline{\tau})$ such that $g \cdot v \cdot g^{-1} \cdot v^{-1} \in V_0$ for every $v \in \xi^{-1}(\overline{V}_1)$.

[3] It is clear that if the topological group (topological ring) $(\overline{G},\ \overline{\tau})$ is Hausdorff then so is $(G,\ \tau)$. In this case without loss of generality $(\widehat{G},\ \widehat{\tau})$ is also assumed to be so, otherwise $(\widehat{G},\ \widehat{\tau})$ is replaced by $(\widehat{G},\ \widehat{\tau})/\widehat{I}$, where $\widehat{I} = [\{e\}]_{(\widehat{G},\ \widehat{\tau})}$.

Proof. The necessity.

1. Let $(\widehat{G},\ \widehat{\tau})$ be a topological group and $\widehat{\xi}:(\widehat{G},\ \widehat{\tau})\to(\overline{G},\ \overline{\tau})$ be the topological homomorphism, mentioned in Definition 2. Since $(G,\ \tau)$ is a subgroup of the topological group $(\widehat{G},\ \widehat{\tau})$ then there exists a neighbourhood $\widehat{V}_0$ of the identity in the topological group $(\widehat{G},\ \widehat{\tau})$ such that $V_0=G\cap\widehat{V}_0$. Since $(\widehat{G},\ \widehat{\tau})$ is a topological group then there exists a neighbourhood of the identity $\widehat{V}_1$ such that $\widehat{V}_1\cdot\widehat{V}_1\cdot\widehat{V}_1^{-1}\subseteq\widehat{V}_0$. Hence $\overline{V}_1=\widehat{\xi}(\widehat{V})$ is a neighbourhood of the identity in $(\overline{G},\ \overline{\tau})$ and $V_1=\widehat{V}_1\cap G$ is a neighbourhood of the identity in $(G,\ \tau)$.

Check that the assertion 1 holds for the neighbourhood $\overline{V}_1$ of the identity in $(\overline{G},\ \overline{\tau})$ and neighbourhood V_1 of the identity in $(G,\ \tau)$.

Indeed, if v is an arbitrary element from $\xi^{-1}(\overline{V}_1)=\xi^{-1}(\widehat{\xi}(\widehat{V}_1))\subseteq\widehat{\xi}^{-1}(\widehat{\xi}(\widehat{V}_1))=\widehat{V}_1\cdot\ker\widehat{\xi}$, then there exist elements $\widehat{g}\in\widehat{V}_1$ and $b\in\ker\widehat{\xi}$ such that $v=\widehat{g}\cdot b$. Hence we obtain taking into account the assertion 3 of Proposition 3 that for every element $g\in\xi^{-1}(\overline{V}_1)\subseteq G$ holds the equality

$$v\cdot g\cdot v^{-1}=(\widehat{g}\cdot b)\cdot g\cdot(\widehat{g}\cdot b)^{-1}=\widehat{g}\cdot(b\cdot g\cdot b^{-1})\cdot\widehat{g}^{-1}=$$

$$\widehat{g}\cdot g\cdot\widehat{g}^{-1}\in\widehat{V}_1\cdot V_1\cdot\widehat{V}_1^{-1}\subseteq\widehat{V}_1\cdot\widehat{V}_1\cdot\widehat{V}_1^{-1}\subseteq\widehat{V}_0.$$

Except that, since G is a normal subgroup in $\widehat{G}$ then $v\cdot g\cdot v^{-1}\in v\cdot G\cdot v^{-1}\subseteq G$ and therefore $v\cdot g\cdot v^{-1}\in\widehat{V}_0\cap G=V_0$. So the assertion 1 holds since elements g and v are supposed to be arbitrary elements.

2. Suppose V_0 to be an arbitrary neighbourhood of the identity in the group $(G,\ \tau)$ and $g\in G$. If $(\widehat{G},\ \widehat{\tau})$ is a topological group and $\widehat{\xi}:(\widehat{G},\ \widehat{\tau})\to(\overline{G},\ \overline{\tau})$ is a topological homomorphism mentioned in the definition 2 then there exists a neighbourhood $\widehat{V}_0$ of the identity in the topological group $(\widehat{G},\ \widehat{\tau})$ such that $V_0=G\cap\widehat{V}_0$. Since $(\widehat{G},\ \widehat{\tau})$ is a topological group then there exists a neighbourhood of the identity $\widehat{V}_1$ in $(\widehat{G},\ \widehat{\tau})$ such that $g\cdot\widehat{V}_1\cdot g^{-1}\cdot\widehat{V}_1^{-1}\subseteq\widehat{V}_0$ and since $\widehat{\xi}:(\widehat{G},\ \widehat{\tau})\to(\overline{G},\ \overline{\tau})$ is a topological homomorphism then $\overline{V}_1=\widehat{\xi}(\widehat{V})$ is a neighbourhood of the identity in $(\overline{G},\ \overline{\tau})$.

Prove that the assertion 2 holds for the neighbourhood of the identity $\overline{V}_1$.

Indeed, let $v\in\xi^{-1}(\overline{V}_1)\subseteq\widehat{\xi}^{-1}(\widehat{\xi}(\widehat{V}_1))=\widehat{V}_1\cdot\ker\widehat{\xi}$. Then there exist elements $\widehat{g}\in\widehat{V}_1$ and $b\in\ker\widehat{\xi}$ such that $v=\widehat{g}\cdot b$. We get, taking into account the assertion 3 of the Proposition 3, that

$$g\cdot v\cdot g^{-1}\cdot v^{-1}=g\cdot(\widehat{g}\cdot b)\cdot g^{-1}\cdot(\widehat{g}\cdot b)^{-1}=g\cdot\widehat{g}\cdot(b\cdot g^{-1}\cdot b^{-1})\cdot\widehat{g}^{-1}=$$

$$g\cdot\widehat{g}\cdot g^{-1}\cdot\widehat{g}^{-1}\in g\cdot\widehat{V}_1\cdot g^{-1}\cdot\widehat{V}_1^{-1}\subseteq\widehat{V}_0.$$

Since $g\cdot v\cdot g^{-1}\cdot v^{-1}\in G$ then $g\cdot v\cdot g^{-1}\cdot v^{-1}\in G\cap\widehat{V}_0=V_0$.

The necessity is completely proved.

The sufficiency.

Let $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ be topological groups and $\xi:(G,\ \tau)\to(\overline{G},\ \overline{\tau})$ be a continuous isomorphism satisfying the assertions 1 and 2.

Write $\widehat{G}$ for the direct product of groups G and $\overline{G}$, i.e. $\widehat{G} = \{(g,\ \overline{g}) \mid g \in G,\ \overline{g} \in \overline{G}\}$. Define a basis of neighbourhoods of the identity $\widehat{\mathcal{B}}$ on $\widehat{G}$ as follows: write $\mathcal{B}$ and $\overline{\mathcal{B}}$ for the sets of all neighbourhoods of the identity in topological groups $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ respectively. Consider the set $\widehat{\mathcal{B}} = \{W(V,\ \overline{V}) \mid V \in \mathcal{B},\ \overline{V} \in \overline{\mathcal{B}}\}$ of subsets $W(V,\ \overline{V}) = \{\big(g \cdot \xi^{-1}(\overline{g}),\ \overline{g}\big) \mid g \in V,\ \overline{g} \in (\overline{V})\}$ of the group $\widehat{G}$.

Check the set $\widehat{\mathcal{B}}$ to be a basis of a certain filter satisfying the assertions (GV_I), (GV_{II}), (GV_{III}) (see [2], p. 14, proposition 1) i.e. that the set $\widehat{\mathcal{B}}$ is a basis of neighbourhoods of zero in a certain group topology $\widehat{\tau}$ on $\widehat{G}$.

Since $(e,\ \overline{e}) = \big(e \cdot \xi^{-1}(\overline{e}),\ \overline{e}\big) \in W(V,\ \overline{V})$ for every $V \in \mathcal{B}$ and $\overline{V} \in \overline{\mathcal{B}}$, then $W(V,\ \overline{V}) \neq \emptyset$, i.e. $\emptyset \notin \widehat{\mathcal{B}}$. Except that if $V,\ U \in \mathcal{B}$ and $\overline{V},\ \overline{U} \in \overline{\mathcal{B}}$ then $V \cap U \in \mathcal{B}$, $\overline{V} \cap \overline{U} \in \overline{\mathcal{B}}$ and $W(V \cap U,\ \overline{V} \cap \overline{U}) \subseteq W(V,\ \overline{V}) \cap W(U,\ \overline{U})$. Hence the set $\widehat{\mathcal{B}}$ is a basis of a certain filter.

Let $W(V_1,\ \overline{V}_1) \in \widehat{\mathcal{B}}$. Since $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ are topological groups then there exists $V_2 \in \mathcal{B}$ and $\overline{V}_2 \in \overline{\mathcal{B}}$ such that $V_2 \cdot V_2 \subseteq V_1$ and $\overline{V}_2 \cdot \overline{V}_2 \subseteq \overline{V}_1$. By the assertion 1 there exists $V_3 \in \mathcal{B}$ and $\overline{V}_3 \in \overline{\mathcal{B}}$, such that $v \cdot V_3 \cdot v^{-1} \subseteq V_2$ for every element $v \in \xi^{-1}(\overline{V}_3)$. Without loss of generality, assume that $V_3 \subseteq V_2$ and $\overline{V}_3 \subseteq \overline{V}_2$.

Prove that $W(V_3,\ \overline{V}_3) \cdot W(V_3,\ \overline{V}_3) \subseteq W(V_1,\ \overline{V}_1)$.

Indeed, let $\big(a \cdot \xi^{-1}(\overline{a}),\ \overline{a}\big) \in W(V_3,\ \overline{V}_3)$ and $\big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big) \in W(V_3,\ \overline{V}_3)$. Hence:

$$\big(a \cdot \xi^{-1}(\overline{a}),\ \overline{a}\big) \cdot \big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big) = \big(a \cdot \xi^{-1}(\overline{a}) \cdot b \cdot \xi^{-1}(\overline{b}),\ \overline{a} \cdot \overline{b}\big),$$

where $\overline{a} \cdot \overline{b} \in \overline{V}_3 \cdot \overline{V}_3 \subseteq \overline{V}_2 \cdot \overline{V}_2 \subseteq \overline{V}_1$ and

$$a \cdot \xi^{-1}(\overline{a}) \cdot b \cdot \xi^{-1}(\overline{b}) = a \cdot \xi^{-1}(\overline{a}) \cdot b \cdot \big(\xi^{-1}(\overline{a})\big)^{-1} \cdot \xi^{-1}(\overline{a}) \cdot \xi^{-1}(\overline{b}) =$$

$$a \cdot \big(\cdot \xi^{-1}(\overline{a}) \cdot b \cdot \xi^{-1}(\overline{a}))^{-1}\big) \cdot \xi^{-1}(\overline{a} \cdot \overline{b}) \in V_3 \cdot V_2 \cdot \xi^{-1}(\overline{a}) \cdot \overline{b})\big) \subseteq V_1 \cdot \xi^{-1}(\overline{a}\overline{b}).$$

Therefore $\big(a \cdot \xi^{-1}(\overline{a}),\ \overline{a}\big) \cdot \big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big) \in W(V_1,\ \overline{V}_1)$. Since $\big(a \cdot \xi^{-1}(\overline{a}),\ \overline{a}\big)$ and $\big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big)$ are arbitrary elements then $W(V_3,\ \overline{V}_3) \cdot W(V_3,\ \overline{V}_3) \subseteq W(V_1,\ \overline{V}_1)$, i.e. the assertion (GV_I) is satisfied.

Let $W(V_1,\ \overline{V}_1) \in \widehat{\mathcal{B}}$. By the assertion 1 there exists $V_2 \in \mathcal{B}$ and $\overline{V}_2 \in \overline{\mathcal{B}}$ such that $v \cdot V_2 \cdot v^{-1} \subseteq V_1$ for every $v \in \xi^{-1}(\overline{V}_2)$. Since $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ are topological groups then there exist $V_3 \in \mathcal{B}$ and $\overline{V}_3 \in \overline{\mathcal{B}}$ such that $V_3^{-1} \subseteq V_2$ and $\overline{V}_3^{-1} \subseteq \overline{V}_1 \cap \overline{V}_2$.

Prove that $\big(W(V_3,\ \overline{V}_3)\big)^{-1} \subseteq W(V_1,\ \overline{V}_1)$.

Indeed, if $\big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big) \in W(V_3 \overline{V}_3)$ then $\big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big)^{-1} = \big((\xi^{-1}(\overline{b}))^{-1} \cdot b^{-1},\ \overline{b}^{-1}\big)$ and $\overline{b}^{-1} \in \overline{V}_3^{-1} \subseteq \overline{V}_1$. Since $\xi^{-1}(\overline{b}^{-1}) \in \xi^{-1}(\overline{V}_3^{-1}) \subseteq \xi^{-1}(\overline{V}_2)$ and $b^{-1} \in V_3^{-1} \subseteq V_2$ then

$$\big(\xi^{-1}(\overline{b})\big)^{-1} \cdot b^{-1} = \big(\xi^{-1}(\overline{b})\big)^{-1} \cdot b^{-1} \cdot \xi^{-1}(\overline{b}) \cdot \big(\xi^{-1}(\overline{b})\big)^{-1} =$$
$$\big(\xi^{-1}(\overline{b}^{-1}) \cdot b^{-1} \cdot (\xi^{-1}(\overline{b}^{-1}))^{-1}\big) \cdot \xi^{-1}(\overline{b}^{-1}) \in V_1 \cdot \xi^{-1}(\overline{b}^{-1}).$$

Hence $\big(b \cdot \xi^{-1}(\overline{b}),\ \overline{b}\big)^{-1} \in W(V_1,\ \overline{V}_1)$.

Since $\big(b\cdot\xi^{-1}(\overline{b}),\ \overline{b}\big)$ is an arbitrary element then

$$\big(W(V_3,\ \overline{V}_3)\big)^{-1}\subseteq W(V_1,\ \overline{V}_1),$$

i.e. the assertion (GV_{II}) is satisfied.

Let $W(V_1,\ \overline{V}_1)\in\widehat{\mathcal{B}}$ and $(g,\ \overline{g})\in\widehat{G}$.

Since $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ are topological groups and ξ is an isomorphism, then there exists $V_2\in\mathcal{B}$ and $\overline{V}_2\in\overline{\mathcal{B}}$ such that $\overline{g}\cdot\overline{V}_2\cdot\overline{g}^{-1}\subseteq\overline{V}_1$. and

$$(g\cdot V_2\cdot g^{-1})\cdot V_2\cdot\Big(\xi^{-1}(\overline{g}^{-1})\cdot V_2\cdot\big(\xi^{-1}(\overline{g}^{-1})\big)^{-1}\Big)\subseteq V_1.$$

By the condition 2 for the neighbourhood $V_2\in\mathcal{B}$ and elements g and $\xi^{-1}(\overline{g}^{-1})\in G$ there exists a neighbourhood $\overline{V}_3\in\overline{\mathcal{B}}$ such that $g\cdot\overline{h}\cdot g^{-1}\overline{h}^{-1}\in V_2$ and

$$\xi^{-1}(\overline{g}^{-1})\cdot h\cdot\big(\xi^{-1}(\overline{g}^{-1})\big)^{-1}\cdot h^{-1}\in V_2$$

for every element $h\in\xi^{-1}(\overline{V}_3)$. Without loss of generality assume $\overline{V}_3^{-1}=\overline{V}_3\subseteq\overline{V}_2$.

Prove that

$$(g,\ \overline{g})\cdot W(V_2,\ \overline{V}_3)\cdot(g,\ \overline{g})^{-1}\subseteq W(V_1,\ \overline{V}_1).$$

Indeed, if $\big(v\cdot\xi^{-1}(\overline{v}),\ \overline{v}\big)\in W(V_2,\ \overline{V}_3)$, then $v\in V_2$ and $\overline{v}\in\overline{V}_3$. Hence

$$(g,\ \overline{g})\cdot\big(v\cdot\xi^{-1}(\overline{v}),\ \overline{v}\big)\cdot(g,\ \overline{g})^{-1}=\big(g\cdot v\cdot\xi^{-1}(\overline{v})\cdot g^{-1},\ \overline{g}\cdot\overline{v}\cdot\overline{g}^{-1}\big)=$$
$$\Big(\big(g\cdot v\cdot\xi^{-1}(\overline{v})\cdot g^{-1}\big)\cdot\big(\xi^{-1}(\overline{g}\cdot\overline{v}^{-1}\cdot\overline{g}^{-1})\big)\cdot\big(\xi^{-1}(\overline{g}\cdot\overline{v}^{-1}\cdot\overline{g}^{-1})\big)^{-1},\ \overline{g}\cdot\overline{v}\cdot\overline{g}^{-1}\Big)=$$
$$\Big(\big(g\cdot v\cdot\xi^{-1}(\overline{v})\cdot g^{-1}\big)\cdot\big(\xi^{-1}(\overline{g}\cdot\overline{v}^{-1}\cdot\overline{g}^{-1})\big)\cdot\big(\xi^{-1}(\overline{g}\cdot\overline{v}\cdot\overline{g}^{-1})\big),\ \overline{g}\cdot\overline{v}\cdot\overline{g}^{-1}\Big)\ \text{ where}$$
$\overline{g}\cdot\overline{v}\cdot\overline{g}^{-1}\in\overline{g}\cdot\overline{V}_3\cdot\overline{g}^{-1}\subseteq\overline{g}\cdot\overline{V}_2\cdot\overline{g}^{-1}\subseteq\overline{V}_1$ and
$$\big(g\cdot v\cdot\xi^{-1}(\overline{v})\cdot g^{-1}\big)\cdot\big(\xi^{-1}(\overline{g}\cdot\overline{v}^{-1}\cdot\overline{g}^{-1})\big)=$$
$$\big(g\cdot v\cdot g^{-1}\cdot g\cdot\xi^{-1}(\overline{v})\cdot g^{-1}\big)\cdot\big(\big(\xi^{-1}(\overline{v})\big)^{-1}\cdot\xi^{-1}(\overline{v})\cdot\xi^{-1}(\overline{g})\cdot\xi^{-1}(\overline{v}^{-1})\cdot\xi^{-1}(\overline{g}^{-1})\big)\in$$
$$(g\cdot V_2\cdot g^{-1})\cdot\big(g\cdot\xi^{-1}(\overline{v})\cdot g^{-1}\cdot(\xi^{-1}(\overline{v}))^{-1}\big)\times$$
$$\big(\xi^{-1}(\overline{v})\cdot\xi^{-1}(\overline{g})\cdot\xi^{-1}(\overline{v}^{-1})\cdot\xi^{-1}(\overline{g}^{-1})\big)\subseteq$$
$$(g\cdot V_2\cdot g^{-1})\cdot V_2\cdot\Big(\xi^{-1}(\overline{g})\cdot\xi^{-1}(\overline{g}^{-1})\cdot\xi^{-1}(\overline{v})\cdot\xi^{-1}(\overline{g})\cdot\xi^{-1}(\overline{v}^{-1})\cdot\xi^{-1}(\overline{g}^{-1})\Big)=$$
$$(g\cdot V_2\cdot g^{-1})\cdot V_2\cdot\Big(\xi^{-1}(\overline{g})\cdot\big(\xi^{-1}(\overline{g}^{-1})\cdot\xi^{-1}(\overline{v})\cdot\xi^{-1}(\overline{g})\big)\cdot\big((\xi^{-1}(\overline{v}))^{-1}\big)\cdot\xi^{-1}(\overline{g}^{-1})\Big)\subseteq$$
$$(g\cdot V_2\cdot g^{-1})\cdot V_2\cdot\big(\xi^{-1}(\overline{g})\cdot V_2\cdot\xi^{-1}(\overline{g}^{-1})\big)\subseteq V_1,$$

since $\xi^{-1}(\overline{v})\in\xi^{-1}(\overline{V}_3)$ (see the definition of the neighbourhood $\overline{V}_3$). Hence

$$(g,\ \overline{g})\cdot(v\cdot\xi^{-1}(\overline{v}),\ \overline{v})\cdot(g,\ \overline{g})^{-1}=$$
$$\Big(\big(g\cdot v\cdot\xi^{-1}(\overline{v})\cdot g^{-1}\big)\cdot\big(\xi^{-1}(\overline{g}\cdot\overline{v}^{-1}\cdot\overline{g}^{-1})\big)\cdot\big(\xi^{-1}(\overline{g}\cdot\overline{v}\overline{g}^{-1})\big),\ \overline{g}\cdot\overline{v}\cdot\overline{g}^{-1}\Big)\in W(V_1,\ \overline{V}_1).$$

Since $(v\cdot\xi^{-1}(\overline{v}),\ \overline{v})$ is an arbitrary element then

$$(g,\ \overline{g})\cdot W(V_2,\ \overline{V}_3)\cdot(g,\ \overline{g})^{-1}\subseteq W(V_1,\ \overline{V}_1),$$

i.e. the assertion (GV_{III}) holds.

Therefore the set $\widehat{\mathcal{B}}$ is a basis of neighbourhoods of the identity in a certain group topology $\widehat{\tau}$ on the group $\widehat{G}$. Prove that the topological group $(\widehat{G},\ \widehat{\tau})$ is the desired one.

One can easily see that $G' = \{(g,\ \overline{e}) \mid g \in G\}$ is a normal subgroup in $\widehat{G}$ and, since $W(V,\ \overline{V}) \cap G' = \{(g,\ \overline{e}) \mid g \in V\}$, then the mapping $\xi' : (G,\ \tau) \to (G',\ \widehat{\tau}\mid_{G'})$ which puts in correspondence the element $(g,\ \overline{e}) \in G'$ to the element $g \in G$ is a topological isomorphism.

Identify the topological group $(G,\ \tau)$ with a subgroup $(G',\ \widehat{\tau}|_{G'})$ of a topological group $(\widehat{G},\ \widehat{\tau})$ with respect to the mapping ξ'.

Note that G' is a normal subgroup in $\widehat{G}$. Hence taking into account the identification given above we get that $\xi(g,\ \overline{e}) = \xi(g)$ and hence the homomorphism $\widehat{\xi} : \widehat{G} \to \overline{G}$ putting $\widehat{\xi}(g,\ \overline{g})$ in correspondence to $\xi(g)$ is an extension of the isomorphism ξ. It remains to check only the homomorphism $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ to be topological, i.e. to be continuous and open.

Let $\overline{V} \in \overline{\mathcal{B}}$. Since $(\overline{G},\ \overline{\tau})$ is a topological group then there exists a neighbourhood $\overline{V}_1 \in \overline{\mathcal{B}}$ such that $\overline{V}_1 \cdot \overline{V}_1 \subseteq \overline{V}$. Since $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a continuous isomorphism then there exists a neighbourhood $V_1 \in \mathcal{B}$, such that $\xi(V_1) \subseteq \overline{V}_1$. Hence

$$\begin{aligned}
&\widehat{\xi}(W(V_1,\ \overline{V}_1)) = \{\widehat{\xi}\big(g \cdot \xi^{-1}(\overline{g}),\ \overline{g}\big) \mid g \in V_1,\ \overline{g} \in \overline{V}_1\} = \\
&\{\widehat{\xi}\big(g \cdot \xi^{-1}(\overline{g})\big) \mid g \in V_1,\ \overline{g} \in \overline{V}_1\} = \\
&\{\xi(g) \cdot \overline{g}) \mid g \in V_1,\ \overline{g} \in \overline{V}_1\} = \xi(V_1) \cdot \overline{V}_1 \subseteq \overline{V}_1 \cdot \overline{V}_1 \subseteq \overline{V},
\end{aligned}$$

and hence $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ is a continuous homomorphism.

Since

$$\widehat{\xi}(W(V,\ \overline{V})) = \{\widehat{\xi}\big(g \cdot \xi^{-1}(\overline{g}),\ \overline{g}\big) \mid g \in V,\ \overline{g} \in \overline{V}\} =$$

$$\{\widehat{\xi}\big(g \cdot \xi^{-1}(\overline{g})\big) \mid g \in V,\ \overline{g} \in \overline{V}\} \supseteq \{\xi(e) \cdot \overline{g}) \mid \overline{g} \in \overline{V}\} = \overline{V}$$

for every neighbourhood $\overline{V} \in \overline{\mathcal{B}}$ then $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\overline{G},\ \overline{\tau})$ is an open homomorphism, that completes the proof of Theorem. □

5 Corollary. *Let $(G,\ \tau)$ be a group equipped with the discrete topology, $(\overline{G},\ \overline{\tau})$ be a topological group and $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ be a continuous isomorphism. The isomorphism ξ is semitopological iff for every element $g \in G$ there exists a neighbourhood $\overline{V}$ of the identity in $(\overline{G},\ \overline{\tau})$ such that $g \cdot \big(\xi^{-1}(\overline{v})\big) = \big(\xi^{-1}(\overline{v})\big) \cdot g$ for every $\overline{v} \in \overline{V}$.*

Proof. Necessity. Indeed, since the topology τ is discrete, then $V_0 = \{e\}$ is a neighbourhood of the identity in $(G,\ \tau)$. If $\overline{V}_1$ is a neighbourhood of the identity in $(\overline{G},\ \overline{\tau})$ such that its elements satisfy the condition 2 of Theorem 4 for the element $g \in G$ and neighbourhood of the identity V_0 in $(G,\ \tau)$, then $g \cdot \xi^{-1}(\overline{v}) \cdot (\xi^{-1}(\overline{v}))^{-1} \cdot g^{-1} = e$ for every element $\overline{v} \in \overline{V}$, which is equivalent to the assertion $g \cdot \xi^{-1}(\overline{v}) = \xi^{-1}(\overline{v}) \cdot g$ for every element $\overline{v} \in \overline{V}$.

Sufficiency. Let V be an arbitrary neighbourhood of the identity in $(G,\ \tau)$. Since $\{e\}$ a neighbourhood of the identity in $(G,\ \tau)$ and $\xi^{-1}(\bar{g}) \cdot e \cdot (\xi^{-1}\bar{g})^{-1} = e \in V$ for every element $\bar{g} \in \bar{G}$ then the condition 1 of Theorem 4 holds.

Except that since for every element $\bar{g} \in \bar{G}$ there exists a neighbourhood $\bar{V}_1$ of the identity in $(\bar{G},\ \bar{\tau})$ such that $g \cdot \xi^{-1}(\bar{v}) = \xi^{-1}(\bar{v}) \cdot g$ for every $\bar{v} \in \bar{V}_1$ then $g \cdot \xi^{-1}(\bar{v}) \cdot g^{-1} \cdot (\xi^{-1}(\bar{v}))^{-1} = e \in V$ for every $\bar{v} \in \bar{V}_1$. Hence the condition 2 Theorem 4 holds. □

6 Corollary. *Let G and $\bar{G}$ be groups and $f : G \to \bar{G}$ be a certain group isomorphism. If $\{\tau_\gamma |\ \gamma \in \Gamma\}$ and $\{\bar{\tau}_\gamma |\ \gamma \in \Gamma\}$ are such families of group topologies on G and $\bar{G}$, respectively, that for every $\gamma \in \Gamma$ the isomorphism $f : (G,\ \tau_\gamma) \to (\bar{G},\ \bar{\tau}_\gamma)$ is semitopological where $\tau = sup\{\tau_\gamma |\ \gamma \in \Gamma\}$ and $\bar{\tau} = sup\{\bar{\tau}_\gamma |\ \gamma \in \Gamma\}$, then so is the isomorphism $f : (G,\ \tau) \to (\bar{G},\ \bar{\tau})$.*

The corollary follows from Theorem 4 and from the outlook of neighbourhoods of the identity in $sup\{\tau_\gamma |\ \gamma \in \Gamma\}$ (see 1.2.22 in [3]).

7 Theorem. *Let $(G,\ \tau)$, $(\overline{G},\ \overline{\tau})$ be topological groups and $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ be a semitopological isomorphism. If A is a subgroup of the group G and $\overline{A} = \xi(A)$, then $\xi\ |_A : (A,\ \tau\ |_A) \to (\overline{A},\ \overline{\tau}\ |_{\overline{A}})$ is a semitopological isomorphism.*

Proof. If U is a neighbourhood of the identity in $(A,\ \tau\ |_A)$ then $U = V \cap A$ for a certain neighbourhood V of the identity in $(G,\ \tau)$. Since $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a semitopological isomorphism then there exist neighbourhoods $\overline{V}$ and V_1 of the identity in $(\overline{G},\ \overline{\tau})$ and $(G,\ \tau)$ respectively such that $v \cdot V_1 \cdot v^{-1} \subseteq V$ for every element $v \in \xi^{-1}(\overline{V})$. Hence $(V_1 \cap A)$ and $\overline{A} \cap \overline{V}$ are neighbourhoods of identities in $(A,\ \tau\ |_A)$ and $(\overline{A},\ \overline{\tau}\ |_{\overline{A}})$ respectively. Note that since $\xi : G \to \overline{G}$ is an isomorphism, then $\xi^{-1}(\overline{A}) = \xi^{-1}(\xi(A)) = A$ and hence $v \cdot (V_1 \cap A) \cdot v^{-1} \subseteq V \cap A = U$ for every $v \in \xi^{-1}\big(\overline{A} \cap \overline{V}\big)$. It means that the assertion 1 of Theorem 4 holds for the isomorphism $\xi\ |_A : (A,\ \tau\ |_A) \to (\overline{A},\ \overline{\tau}\ |_{\overline{A}})$.

Let $g \in A$ and U be a neighbourhood of the identity in $(A,\ \tau\ |_A)$. Hence $U = V \cap A$ for a certain neighbourhood V of the identity in $(G,\ \tau)$. Since $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ is a semitopological isomorphism then for a neighbourhood V of the identity in topological group $(G,\ \tau)$ and for the element $g \in A \subseteq G$ there exists a neighbourhood $\overline{V}_1$ of the identity in $(\overline{G},\ \overline{\tau})$ such that $g \cdot v \cdot g^{-1} \cdot v^{-1} \in V$ for every $v \in \xi^{-1}(\overline{V}_1)$. Since $\xi^{-1}(\overline{A}) = \xi^{-1}(\xi(A)) = A$ then $g \cdot v \cdot g^{-1} \cdot v^{-1} \in V \cap A = U$ for every $v \in \xi^{-1}(\overline{V}_1 \cap \overline{A})$, i.e. the assertion 2 of Theorem 4 holds for the isomorphism $\xi\ |_A : (A,\ \tau\ |_A) \to (\overline{A},\ \overline{\tau}\ |_{\overline{A}})$. Hence $\xi\ |_A : (A,\ \tau\ |_A) \to (\overline{A},\ \overline{\tau}\ |_{\overline{A}})$ is a semitopological isomorphism. □

8 Theorem. *Let $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ be topological groups and $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$ be a semitopological isomorphism. If A is a normal subgroup of the group G and*

$$\begin{array}{ccc} G & \xrightarrow{\ \xi\ } & \overline{G} \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\overline{\eta}} \\ G/A & \xrightarrow{\ \widehat{\xi}\ } & \overline{G}/\xi(A) \end{array}$$

where:

$\eta : G \to G/A$ *is the canonical homomorphism (i.e.* $\eta(g) = g \cdot A$*);*

$\overline{\eta} : \overline{G} \to \overline{G}/(\xi(A))$ *is the canonical homomorphism (i.e.* $\overline{\eta}(\overline{g}) = \overline{g} \cdot \xi(A)$*);*

$\widehat{\xi} : G/A \to \overline{G}/(\xi(A))$ *is the canonical isomorphism (i.e.* $\widehat{\xi}(g \cdot A) = \xi(g) \cdot \xi(A)$*).*

Hence $\widehat{\xi} : (G,\ \tau)/A \to (\overline{G},\ \overline{\tau})/(\xi(A))$ *is a semitopological isomorphism.*

Proof. If $\widehat{V}_0$ is a neighbourhood of the identity in $(G,\ \tau)/A$ then $V_0 = \eta^{-1}(\widehat{V}_0)$ is a neighbourhood of the identity in $(G,\ \tau)$. By the assertion 1 of Theorem 4 there exist neighbourhoods V_1 and $\overline{V}_1$ of the identity in $(G,\ \tau)$ and $(\overline{G},\ \overline{\tau})$ respectively such that $v \cdot V_1 \cdot v^{-1} \subseteq V_0$ for every $v \in \xi^{-1}(\overline{V}_1)$. Hence $\widehat{V}_1 = \eta(V_1)$ and $\widetilde{V_1} = \overline{\eta}(\overline{V}_1)$ are neighbourhoods of the identity in $(G,\ \tau)/A$ and $(\overline{G},\ \overline{\tau})/\xi(A)$, respectively.

Note that $\widehat{\xi}\big(\eta(\xi^{-1}(\overline{V}_1))\big) = \overline{\eta}(\overline{V}_1) = \widetilde{V_1}$. Hence if $\widehat{v} \in \widehat{\xi}^{-1}(\widetilde{V_1})$ then there exists an element $v \in \xi^{-1}(\overline{V}_1)$ such that $\eta(v) = \widehat{v}$. Hence

$$\widehat{v} \cdot \widehat{V}_1 \cdot \widehat{v}^{-1} = \eta(v) \cdot \eta(V_1) \cdot (\eta(v))^{-1} = \eta(v \cdot V_1 \cdot v^{-1}) \subseteq \eta(V_0) = \eta\big(\eta^{-1}(\widehat{V}_0)\big) = \widehat{V}_0.$$

Hence the assertion 1 of Theorem 4 holds for the isomorphism $\widehat{\xi} : (G,\ \tau)/A \to (\overline{G},\ \overline{\tau})/(\xi(A))$.

Check the assertion 2 of Theorem 4 to hold for the isomorphism $\widehat{\xi} : (G,\ \tau)/A \to (\overline{G},\ \overline{\tau})/(\xi(A))$.

Let $\widetilde{g} \in (\overline{G},\ \overline{\tau})/(\xi(A))$ and $\widehat{V}$ be a neighbourhood of the identity in $(G,\ \tau)/A$. Hence $V = \eta^{-1}(\widehat{V})$ is a neighbourhood of the identity in $(G,\ \tau)$ and there exists an element $g \in G$ such that $\widehat{g} = \eta(g)$. Since the assertion 2 of Theorem 4 holds for the homomorphism $\xi : (G,\ \tau) \to (\overline{G},\ \overline{\tau})$, then there exists such a neighbourhood $\overline{V}_1$ of the identity in $(\overline{G},\ \overline{\tau})$ that $g \cdot v \cdot g^{-1} \cdot v^{-1} \in V$ for every $v \in \xi^{-1}(\overline{V}_1)$. Hence $\widetilde{V_1} = \overline{\eta}(\overline{V}_1)$ is a neighbourhood of the identity in $(\overline{G},\ \overline{\tau})/(\xi(A))$. Note that $\eta\big(\xi^{-1}(\overline{V}_1)\big) = \widehat{\xi}^{-1}(\overline{\eta}(\overline{V}_1)) = \widehat{\xi}^{-1}(\widetilde{V}_1)$.

If $\widehat{v} \in \widehat{\xi}^{-1}(\widetilde{V}_1))$, then there exists an element $v \in \xi^{-1}(\overline{V}_1)$ such that $\eta(v) = \widehat{v}$. Hence

$$\widehat{g} \cdot \widehat{v} \cdot \widehat{g}^{-1} \cdot \widehat{v}^{-1} = \eta(g) \cdot \eta(v) \cdot \big(\eta(g)\big)^{-1} \cdot \big(\eta(v)\big)^{-1} = \eta(g \cdot v \cdot g^{-1} \cdot v^{-1}) \in \eta(V) = \widehat{V},$$

i.e. the assertion 2 of Theorem 4 holds for the isomorphism $\widehat{\xi} : (G,\ \tau)/A \to (\overline{G},\ \overline{\tau})/(\xi(A))$. The theorem is completely proved. □

9 Theorem. *Let* $\{(G_\gamma,\ \tau_\gamma) \mid \gamma \in \Gamma\}$ *and* $\{(\overline{G}_\gamma,\ \overline{\tau}_\gamma) \mid \gamma \in \Gamma\}$ *be two families of topological groups and for every* $\gamma \in \Gamma$ *there exists a semitopological isomorphism* $\xi_\gamma : (G_\gamma,\ \tau_\gamma) \to (\overline{G}_\gamma,\ \overline{\tau}_\gamma)$*. If* $(\widehat{G},\ \widehat{\tau}) = \prod\limits_{\gamma\in\Gamma}(G_\gamma,\ \tau_\gamma)$ *and* $(\widetilde{G},\ \widetilde{\tau}) = \prod\limits_{\gamma\in\Gamma}(\overline{G}_\gamma,\ \overline{\tau}_\gamma)$ *are direct products of these families equipped with the Tychonoff topology and* $\widehat{\xi} : \widehat{G} \to \widetilde{G}$ *is a canonical isomorphism (i.e.* $\xi_\gamma(\mathrm{pr}_\gamma(\widehat{g})) = \mathrm{pr}_\gamma\widehat{\xi}(\widehat{g})$*) for any* $\gamma \in \Gamma$*, then* $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\widetilde{G},\ \widetilde{\tau})$ *is a semitopological isomorphism.*

Proof. If $\widehat{V}$ is a neighbourhood of the identity in $(\widehat{G},\ \widehat{\tau})$, then there exists a finite subset $S \subseteq \Gamma$ such that for every $\gamma \in S$ there exists a neighbourhood V_γ in $(G_\gamma,\ \tau_\gamma)$

such that $\bigcap\limits_{\gamma\in S} \mathrm{pr}_\gamma{}^{-1}(V_\gamma) \subseteq \widehat{V}$. Since for every $\gamma \in \Gamma$ the mapping $\xi_\gamma : (G_\gamma,\ \tau_\gamma) \to (\overline{G}_\gamma,\ \overline{\tau}_\gamma)$ is a semitopological isomorphism then there exist neighbourhoods $\overline{V}_\gamma$ and U_γ of the identity in $(\overline{G}_\gamma,\ \overline{\tau}_\gamma)$ and $(G_\gamma,\ \tau_\gamma)$, respectively such that $v_\gamma \cdot U_\gamma \cdot v_\gamma^{-1} \subseteq V_\gamma$ for every elements $v_\gamma \in \xi^{-1}(\overline{V}_\gamma)$. Hence $\widetilde{V} = \bigcap\limits_{\gamma\in S} \mathrm{pr}_\gamma^{-1}(V_\gamma)$ and $\widehat{U} = \bigcap\limits_{\gamma\in S} \mathrm{pr}_\gamma^{-1}(U_\gamma)$ are neighbourhoods of the identity in $(\widetilde{G},\ \widetilde{\tau})$ and $(\widehat{G},\ \widehat{\tau})$ respectively.

If $\widehat{v} \in \widehat{\xi}^{-1}(\widetilde{V})$ then $\mathrm{pr}_\gamma(\widehat{v}) \in \xi_\gamma^{-1}(\overline{V}_\gamma)$ and therefore $\mathrm{pr}_\gamma(\widehat{v}) \cdot U_\gamma \cdot (\mathrm{pr}_\gamma(\widehat{v}))^{-1} \subseteq V_\gamma$ for every $\gamma \in S$. Hence $\widehat{v} \cdot \widehat{U} \cdot \widehat{v}^{-1} \subseteq \widehat{V}$, i.e. the condition 1 of Theorem 4 holds for the isomorphism $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\widetilde{G},\ \widetilde{\tau})$.

If $\widehat{g} \in \widehat{G}$ and $\widehat{V}$ is a neighbourhood of the identity in $(\widehat{G},\ \widehat{\tau})$, then there exists a finite set $S \subseteq \Gamma$ and for every $\gamma \in S$ there exists a neighbourhood V_γ of the identity in $(G_\gamma,\ \tau_\gamma)$ such that $\bigcap\limits_{\gamma\in S} \mathrm{pr}_\gamma^{-1}(V_\gamma) \subseteq \widehat{V}$. Since for every $\gamma \in \Gamma$ the mapping $\xi_\gamma : (G_\gamma,\ \tau_\gamma) \to (\overline{G}_\gamma,\ \overline{\tau}_\gamma)$ is a semitopological isomorphism then for the neighbourhood V_γ of the identity in the topological group $(G_\gamma,\ \tau_\gamma)$ and for the element $g_\gamma = \mathrm{pr}_\gamma(\widehat{g})$ there exists a neighbourhood $\overline{V}_\gamma$ of the identity in $(\overline{G}_\gamma,\ \overline{\tau}_\gamma)$ such that $g_\gamma \cdot v_\gamma \cdot g_\gamma^{-1} \cdot v_\gamma^{-1} \in V_\gamma$ for every $v_\gamma \in \xi_\gamma^{-1}(\overline{V}_\gamma)$. Hence $\widetilde{V} = \bigcap\limits_{\gamma\in S} \mathrm{pr}_\gamma{}^{-1}(\overline{V}_\gamma)$ is a neighbourhood of the identity in $(\widetilde{G},\ \widetilde{\tau})$ and if $\widehat{v} \in \widehat{\xi}^{-1}(\widetilde{V})$ then $v_\gamma = \mathrm{pr}_\gamma(\widehat{v}) \in \xi_\gamma^{-1}(\mathrm{pr}_\gamma(\widetilde{V})) = \xi_\gamma^{-1}(\mathrm{pr}_\gamma(\overline{V}_\gamma))$ for any $\gamma \in S$. Hence $\widehat{g} \cdot \widehat{v} \cdot \widehat{g}^{-1} \cdot \widehat{v}^{-1} \in \bigcap\limits_{\gamma\in S} \mathrm{pr}_\gamma^{-1}(V_\gamma) \subseteq \widehat{V}$, i.e. the assertion 2 of Theorem 4 holds for the isomorphism $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\widetilde{G},\ \widetilde{\tau})$.

Hence $\widehat{\xi} : (\widehat{G},\ \widehat{\tau}) \to (\widetilde{G},\ \widetilde{\tau})$ is a semitopological isomorphism. □

10 Remark. *Theorem 9 remains valid if groups* $\widehat{G} = \prod\limits_{\gamma\in\Gamma} G_\gamma$ *and* $\widetilde{G} = \prod\limits_{\gamma\in\Gamma} \overline{G}_\gamma$ *are equipped not with the Tychonoff topology but with the topology of* $\mathfrak{m}$*-product (see [3], Definition 4.1.3).*

11 Remark. *The following example proves that the superposition of semitopological isomorphisms needs not to be semitopological. The topological groups mentioned in it are not Hausdorff. An example with Hausdorff topological groups can be obtained by an easy modification of the given one.*

12 Example. *Let G be a nilpotent group of index 2 (i.e. G a noncommutative group such that its quotient group G/Z by its center Z is commutative). Consider the following three group topologies on the group G:*

τ_0 is the discrete topology, i.e. the set $\{\{e\}\}$ is a basis of neighbourhoods of the identity in $(G,\ \tau_0)$;

τ_1 is the topology such that the set $\{Z\}$ is a basis of neighbourhoods of the identity in $(G,\ \tau_1)$;

τ_2 is the antidiscrete topology, i.e. the set $\{G\}$ is a basis of neighbourhoods of the identity in $(G,\ \tau_2)$;

Let $\xi : G \to G$ be an identity mapping. One can easily see that assertions 1 and 2 of Theorem 4 hold for the continuous isomorphisms $\xi : (G,\ \tau_0) \to (G,\ \tau_1)$ and $\xi : (G,\ \tau_1) \to (G,\ \tau_2)$ and hence they are semitopological.

Prove now that the assertion 2 of Theorem 4 does not hold for the isomorphism $\xi : (G,\ \tau_0) \to (G,\ \tau_2)$, *i.e. it is not semitopological.*

Suppose the contrary, i.e. the assertion 2 of Theorem 4 holds for the isomorphism $\xi : (G,\ \tau_0) \to (G,\ \tau_2)$. *Since the group* G *is not a commutative group, then there exist elements* $g,\ v \in G$ *such that* $g \cdot v \neq v \cdot g$, *i.e.* $g \cdot v \cdot g^{-1} \cdot v^{-1} \neq e$. *Then for an element* $g \in G$ *and the neighbourhood* $\{e\}$ *of the identity in* $(G,\ \tau_0)$ *there exists a neighbourhood* V *of the identity in* $(G,\ \tau_2)$ *such that* $g \cdot u \cdot g^{-1} \cdot u^{-1} \in \{e\}$ *for every element* $u \in \xi^{-1}(V)$. *Since* τ_2 *is the antidiscrete topology then* $V = G$ *and hence we may assume the element* u *to be equal to* v. *Hence* $g \cdot v \cdot v^{-1} \cdot g^{-1} \in \{e\}$ *that contradicts to the choice of elements* $g,\ v \in G$.

13 Problem. *Given a class* $\mathfrak{K}$ *of topological groups (rings) and a group (ring)* G. *What is the group (ring) topology* τ *on* G *such that* $(G,\ \tau) \in \mathfrak{K}$ *and every semitopological homomorphism* $(G,\ \tau) \to (H,\ \mu)$ *is topological, where* $(H,\ \mu) \in \mathfrak{K}$ *(so are known to be the topological rings with no generalized zero divisors, see [1], Theorem 2).*

14 Problem. *What is the group (ring)* G *such that for every group (ring) topology* τ *on it every semitopological isomorphism* $(G,\ \tau) \to (H,\ \mu)$ *is topological (so are known to be the rings with an identity).*

15 Problem. *What are the continuous isomorphisms which are superpositions of semitopological (note that they need not to be semitopological, see the example 12).*

Author does not know whether every continuous isomorphism of topological groups is so.

16 Problem. *Let* G *and* $\overline{G}$ *be groups,* $f : G \to \overline{G}$ *be an isomorphism,* $\{\tau_\gamma \mid \gamma \in \Gamma\}$ *and* $\{\overline{\tau}_\gamma \mid \gamma \in \Gamma\}$ *be families of group topologies on* G *and* $\overline{G}$ *respectively such that* $f : (G, \tau_\gamma) \to (G, \overline{\tau}_\gamma)$ *is a semitopological isomorphism for every* $\gamma \in \Gamma$. *Write* τ *for* $\inf\{\tau_\gamma \mid \gamma \in \Gamma\}$ *and* $\overline{\tau}$ *for* $\inf\{\overline{\tau}_\gamma \mid \gamma \in \Gamma\}$. *Is the isomorphism* $f : (G, \tau) \to (\overline{G}, \overline{\tau})$ *semitopological?*

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Received February 20, 2004