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The classification of $GL(2,R)$-orbits' dimensions for system $s(0,2)$ and the factorsystem $s(0,1,2)/GL(2,R)$ *

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Abstract. Two-dimensional systems of two autonomous polynomial differential equations with homogeneities of the zero, first and second orders are considered with respect to the group of center-affine transformations $GL(2,R)$. The problem of the classification of $GL(2,R)$-orbits' dimensions is solved completely for system $s(0,2)$ with the help of Lie algebra of operators corresponding to $GL(2,R)$ group, and algebras of invariants and comitants. A factorsystem $s(0,1,2)/GL(2,R)$ for system $s(0,1,2)$ is built and with its help two invariant $GL(2,R)$-integrals are obtained for the system $s(1,2)$ in some necessary conditions for the existence of singular point of the type "center".

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Consider the real system of differential equations

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta}x^\alpha x^\beta, \quad (j,\alpha,\beta = 1,2), \tag{1}$$

which will be denoted by $s(0,1,2)$, where the coefficient tensor $a^j_{\alpha\beta}$ is symmetrical in lower indexes, in which the complete convolution takes place, and the group of center-affine transformations $GL(2,R)$, given by the equalities $\bar{x}^r = q^r_j x^j$, $\Delta_q = = det(q^r_j) \neq 0, (r,j = 1,2)$.

Consider the invariants and comitants of the system (1) with respect to the group $GL(2,R)$, found in [1], which will be used further:

$$K_1 = a^\alpha_{\alpha\beta}x^\beta,\ K_2 = a^p_\alpha x^\alpha x^q \varepsilon_{pq},\ K_5 = a^p_{\alpha\beta}x^\alpha x^\beta x^q \varepsilon_{pq},\ K_6 = a^\alpha_{\alpha\beta}a^\beta_{\gamma\delta}x^\gamma x^\delta,$$

$$K_7 = a^\alpha_{\beta\gamma}a^\beta_{\alpha\delta}x^\gamma x^\delta,\ K_9 = a^\alpha_{p\alpha}a^\beta_{q\gamma}a^\gamma_{\beta\delta}x^\delta\varepsilon^{pq},\ K_{21} = a^p x^q \varepsilon_{pq},\ K_{23} = a^p a^q_{\alpha\beta}x^\alpha x^\beta \varepsilon_{pq},$$

$$K_{25} = a^\alpha a^\beta a^p_{\alpha\beta}x^q\varepsilon_{pq},\ I_1 = a^\alpha_\alpha,\ I_2 = a^\alpha_\beta a^\beta_\alpha,\ I_4 = a^\alpha_p a^\beta_{\beta q}a^\gamma_{\alpha\gamma}\varepsilon^{pq},\ I_5 = a^\alpha_p a^\beta_{\gamma q}a^\gamma_{\alpha\beta}\varepsilon^{pq},$$

$$I_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{\alpha q}a^\delta_{\beta\delta}\varepsilon^{pq},\ I_7 = a^\alpha_{pr}a^\beta_{q\alpha}a^\gamma_{s\beta}a^\delta_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs},\ I_8 = a^\alpha_{pr}a^\beta_{q\alpha}a^\gamma_{s\delta}a^\delta_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},$$

$$I_9 = a^\alpha_{pr}a^\beta_{q\beta}a^\gamma_{s\gamma}a^\delta_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs},\ I_{13} = a^\alpha_p a^\beta_{qr}a^\gamma_{\gamma s}a^\delta_{\alpha\beta}a^\mu_{\delta\mu}\varepsilon^{pq}\varepsilon^{rs},$$

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$$I_{15} = a^{\alpha}_{pr} a^{\beta}_{qk} a^{\gamma}_{\alpha s} a^{\delta}_{\delta l} a^{\mu}_{\beta\gamma} a^{\nu}_{\mu\nu} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{kl},\ I_{17} = a^{\alpha} a^{\beta}_{\alpha\beta},\ I_{25} = a^{\alpha} a^{\beta}_{\beta p} a^{\gamma}_{\delta q} a^{\delta}_{\alpha\gamma} \varepsilon^{pq}. \qquad (2)$$

where ε^{pq} and ε_{pq} are unit bivectors ($\varepsilon^{11} = \varepsilon^{22} = 0,\ \varepsilon^{12} = -\varepsilon^{21} = 1,\ \varepsilon_{11} = \varepsilon_{22} = 0,$ $\varepsilon_{12} = -\varepsilon_{21} = 1$).

Remark 1. For $I_1 = 0, K_2 \equiv 0$ the system (1) takes the form (it will be denoted by s(0,2) further)

$$\frac{dx^j}{dt} = a^j + a^j_{\alpha\beta} x^{\alpha} x^{\beta}, \quad (j, \alpha, \beta = 1, 2). \qquad (3)$$

I. The proof of the next theorem is based on the classification of $GL(2,R)$-orbits' dimensions for system $s(2)$ from [2]:

Theorem 1. *If $I_1 = 0, K_2 \equiv 0$, the $GL(2,R)$-orbit of the system (3) has the dimension*

4 for $K_1 K_5 \not\equiv 0,\ F_1 + K_9 + \beta \not\equiv 0$, *or*
$K_5 \not\equiv 0,\ K_1 \equiv 0,\ F_2 + K_9 + \beta \not\equiv 0$;

3 for $K_1 K_5 \not\equiv 0,\ F_1 + K_9 + \beta \equiv 0$, *or*
$K_5 \not\equiv 0,\ K_1 \equiv 0,\ F_2 + K_9 + \beta \equiv 0,\ K_7 + K_{21} \not\equiv 0$, *or*
$K_5 \equiv 0,\ K_1 K_{21} \not\equiv 0$;

2 for $K_{21} \equiv 0, K_1 + K_5 \not\equiv 0, K_5(K_1 + K_7) \equiv 0$, *or*
$K_5 \equiv 0, K_1 K_{21} \equiv 0, K_1^2 + K_{21}^2 \not\equiv 0$;

0 for $K_1 \equiv K_5 \equiv K_{21} \equiv 0$,

where $\beta = 27 I_8 - I_9 - 18 I_7$, $F_1 = K_5[-2 I_{17} K_5 + K_1(2 K_1 K_{21} - 3 K_{23})]$, $F_2 = K_{21}^2 (3 K_1^2 - 2 K_6 - 3 K_7) + 2 K_5 K_{25}$, *and* K_1, K_5, K_6, K_7, K_9, K_{21}, K_{23}, I_7, I_8, I_9, I_{17} *are taken from (2).*

For the system $s(0,3)$ the similar problem was considered in [3]. Remark that in (51) only the sets M_1, $M_4 - M_6$, $M_8 - M_{13}$ should be considered as $GL(2,R)$-invariant nonintersecting sets.

II. According to [4] the classification of $GL(2,R)$-orbits' dimensions could be considered as a division of the set $E^{14}(x,a)$ of the coefficients and variables of the system (1) into invariant manifolds, and the maximal dimension orbit is a nonsingular invariant manifold of the $GL(2,R)$ group.

Remark 2. The condition $K_1 K_5 K_9 \not\equiv 0$ follows from the condition $I_9(I_9 - I_7) \neq 0$, both of them define nonsingular invariant manifolds (see definition in [4]).

The proof is based on the facts that $Rez(K_1, K_5) = I_9$ and $Rez(K_1, K_9) = I_9 - I_7$.

Theorem 2. *On the nonsingular invariant manifold $I_9(I_9 - I_7) \neq 0$ the system (1) has the following factorsystem (see [4]) $s(0,1,2)/GL(2,R)$*

$$\dot{\bar{x}} = I_{17} + \left[\frac{1}{2} I_1 + \frac{-I_1 I_7 - 2 I_{13}}{2 I_9} - \frac{I_4 I_{15}}{I_9 (I_9 - I_7)}\right] \bar{x} - \frac{I_4}{|I_9 - I_7|^{1/2}} \bar{y} +$$

$$+\left[\frac{I_7+I_9}{2I_9}+\frac{I_{15}^2}{I_9(I_9-I_7)^2}\right]\bar{x}^2+2\frac{I_{15}}{|I_9-I_7|^{3/2}}\bar{x}\bar{y}+\frac{I_9}{(I_9-I_7)}\bar{y}^2,$$

$$\dot{\bar{y}}=\frac{I_{25}}{|I_9-I_7|^{1/2}}+\frac{1}{|I_9-I_7|^{1/2}}\left[\frac{I_4I_{15}^2}{I_9^2|I_9-I_7|^2}-\frac{I_4(I_7^2+I_9^2)}{2I_9^2}+I_5\right]\bar{x}+$$

$$+\left[\frac{1}{2}I_1+\frac{I_1I_7+2I_{13}}{2I_9}+\frac{I_4I_{15}}{I_9(I_9-I_7)}\right]\bar{y}-\frac{I_{15}(I_7+I_9)}{2I_9^2|I_9-I_7|^{1/2}}\bar{x}^2-$$

$$-\frac{I_{15}^3}{I_9^2|I_9-I_7|^{3/2}}\bar{x}^2+2\left[\frac{I_9-I_7}{2I_9}-\frac{I_{15}^2}{I_9(I_9-I_7)^2}\right]\bar{x}\bar{y}-\frac{I_{15}}{|I_9-I_7|^{3/2}}\bar{y}^2, \tag{4}$$

for which $K_1=\bar{x}, K_9=\bar{y}$, and K_1, K_9, I_1, I_4, I_5, I_7, I_9, I_{13}, I_{15}, I_{17}, I_{25} are taken from (2).

III. Consider the center conditions from [5] for the system (1) with $a^j=0$ $(j=1,2)$:

$$I_2<0,\ I_1=I_6=I_{13}=0,\ I_4\neq 0. \tag{5}$$

Taking into account the last four conditions from (5) and $I_{17}=I_{25}=0$, and the syzygies from [6], we conclude that the factorsystem (4) will take the form

$$\dot{\bar{x}}=-\frac{I_4}{|I_9-I_7|^{1/2}}\bar{y}+\frac{I_7+I_9}{2I_9}\bar{x}^2+\frac{I_9}{I_9-I_7}\bar{y}^2,$$

$$\dot{\bar{y}}=\frac{1}{|I_9-I_7|^{1/2}}\left[I_9-\frac{I_4(I_7^2+I_9^2)}{2I_9^2}\right]\bar{x}+\frac{I_9-I_7}{I_9}\bar{x}\bar{y}, \tag{6}$$

for which $I_9(I_9-I_7)\neq 0$. We obtain with the help of (6)

Proposition 1. *The system (1) has the following two invariant $GL(2,R)$-integrals on the nonsingular invariant $GL(2,R)$-manifold $I_9(I_9-I_7)\neq 0$ for $I_{17}=I_{25}=0$ and for necessary center conditions $I_1=I_6=I_{13}=0,\ I_4\neq 0$*

$$\mathcal{F}_1\equiv 2I_5I_9^2-I_4(I_7^2+I_9^2)+2I_9(I_9-I_7)K_9=0,$$

$$\mathcal{F}_2\equiv I_7(I_9+I_7)[(I_9-I_7)^2(I_9-3I_7)K_1^2-2I_9^2K_9^2]+[I_5I_9^2+I_4I_7(-2I_9+I_7)]\cdot$$

$$\cdot[-2I_5I_9^2+I_4(I_7^2+I_9^2)-2I_9(I_9+I_7)K_9]=0.$$

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