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Artinal special Lie superalgebras

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Abstract. The artinian special Lie superalgebras are studied in the paper. It is proved, that the gr-prime radical of a artinian special Lie superalgebra is solvable.

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All algebras are supposed to be algebras over a field F.

In 1963 V. Latyshev defined special Lie algebra [1].

We say that an algebra L is called a special Lie algebra or SPI-algebra, if there exists an associative PI-algebra A such that L is included in $A^{(-)}$ as a Lie algebra, where $A^{(-)}$ is a Lie algebra with respect to the operation of commutation $[x, y] = xy - yx$.

The structural theory of special Lie algebras was studied in [2-8] and others.

Let L be a Lie algebra, $a \in L$. By ad_a we shall denote the linear transformation $\mathrm{ad}_a : L \longrightarrow L$, defined by the formula $(x)\mathrm{ad}_a = [x, a]$. We shall denote by $\mathrm{Ad}(L)$ the associative algebra generated in $\mathrm{End}(L)$ by the set $\{\mathrm{ad}_a \mid a \in L\}$.

We say that the algebra is prime if the following assertion holds for every its ideals U, V: if $UV = 0$ then either $U = 0$ or $V = 0$. The definition is given similarly for associative and Lie algebras.

We say that the ideal P of an algebra L is prime if the factor-algebra L/P is prime.

We define the prime radical $P(L)$ as the intersection of all prime ideals of a Lie algebra L.

It was proved in [6] that the prime radical of a special Lie algebra is locally soluble.

As it is impossible to construct a good structural theory for all special Lie algebras, it is necessary to investigate classes of special Lie algebras, for which such a theory exists. For associative algebras there is a good theory for artinian algebras.

By analogy to associative algebras we say that a Lie algebra is artinian if every non-empty descending chain of its ideals is stabilized.

We remark that unlike associative algebras, for which right or left ideals are considered, for Lie algebras there is no necessity to speak about right artinian or left artinian algebras.

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The following theorem was obtained in [9].

Theorem 1. *Let L be an artinian special Lie algebra and $P(L)$ be its prime radical. Then the ideal $P(L)$ is soluble.*

This theorem is true also for Lie superalgebras.

Lie superalgebra L over a field F is a $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ over the field F on which the bilinear operation $[x, y]$ is defined and for homogeneous components the following identities are valid

$$\alpha([x,y]) = \alpha(x) + \alpha(y),$$
$$[x,y] = (-1)^{\alpha(x)\alpha(y)+1}[y,x],$$
$$[x,[y,z]](-1)^{\alpha(x)\alpha(z)} + [z,[x,y]](-1)^{\alpha(-z)\alpha(y)} + [y,[z,x]](-1)^{\alpha(y)\alpha(x)} = 0,$$

where $\alpha(x)$ is the number of the homogeneous component [10].

We call a $\mathbb{Z}_2$-graded associative algebra an associative superalgebra.

The ideal I of an associative superalgebra or Lie superalgebra is called graded if $I = I_0 \oplus I_1$, where $I_i = I \cap A_i, i = 0, 1$. The factor-algebra by a graded two-sided ideal is an associative superalgebra or Lie superalgebra respectively.

We shall use also the concept of graded modulus over associative superalgebra or Lie superalgebra.

We say that an associative superalgebra A is a PI-superalgebra if it satisfies to polynomial identity as an algebra without graduation. It is known that a sufficient condition of the fulfilment of identity in the graded by finite group associative algebra is the fulfilment of identity in a unit component of algebra [11]. In [12] the estimate of the degree of such identity was given.

We call a Lie superalgebra L over a field F special if there exists an associative PI-superslgebra A such that $L \subseteq [A]$, where $[A]$ is the algebra A in respect to operation of the commutation, defined on homogeneous components by the formula

$$[x,y] = xy - (-1)^{\alpha(x)\alpha(y)}yx,$$

where $\alpha(x)$ is the number of the homogeneous component. The associative superalgebra A with respect to the relation to this operation of commutation a is Lie superalgebra $[A]$. The concept of special color Lie superalgebras was studied in [13].

Let L be a Lie superalgebra, $a \in L$. By ad_a we shall denote the linear transformation $\mathrm{ad}_a : L \longrightarrow L$, defined by the formula $x\mathrm{ad}_a = [x, a]$. We shall denote by $\mathrm{Ad}(L)$ the associative algebra generated in $\mathrm{End}(L)$ by set $\{\mathrm{ad}_a \mid a \in L\}$.

The algebra $\mathrm{Ad}(L)$ is an associative superalgebra.

The definition of the solubility is given for Lie superalgebras in the same way as for Lie algebras.

We shall define for Lie superalgebras the concept of a prime algebra and a prime graded ideal in the same way as for associative algebras.

We shall call a gr-prime radical of special Lie superalgebra the intersection of all it gr-prime graded ideals.

For special Lie superalgebras the following result is proved.

Theorem 2. *Let L be an artinian special Lie superalgebra and $P_{gr}(L)$ be its its gr-prime radical. Then the ideal $P_{gr}(L)$ is soluble.*

Proof. We shall consider the sequence of commutators

$$P = P_{gr}(L),\ P^{(1)} = P,\ P^{(2)} = [P^{(1)},\ P^{(1)}], \dots, P^{(k+1)} = [P^{(k)},\ P^{(k)}] \dots$$

Then the inclusions take place $P^{(1)} \supseteq P^{(2)} \supseteq, \dots, \supseteq P^{(k)} \supseteq, \dots$ It is well known [14], that all commutators $P^{(k)}$ are quite characteristic subalgebras and, hence, are ideals of superalgebra L.

As the Lie superalgebra L is artinian then $R^{(m)} = R^{(m+1)}$ for some natural m. We want to prove, that $R^{(k)} = 0$ for some k.

Let d be the degree of polynomial identity in algebra $\mathrm{Ad}(L)$, which exists according to [15].

Let $n = \max(m, [d/2]^2)$.

We shall denote by W the centrelizer of $P^{(n)}$, i.e.

$$W = \{x \mid x \in L, [x, P^{(n)}] = 0\}.$$

In the book [16] it was proved that the centre of an associative superalgebra A graded by an abelian group is graded. It is possible to prove the same is true for the center and the centrelizer of a graded ideal of a Lie superalgebra.

The set W is the graded ideal.

If $W \supseteq P^{(n)}$ then $P^{(n+1)} = 0$. The theorem is proved.

Let's assume that W does not contain $P^{(n)}$. Then we will obtain the contradiction.

We shall consider the factor-superalgebra $\bar{L} = L/W$. By the natural homomorphism $L \to \bar{L}$ the ideal $P^{(n)}$ is mapped to $\bar{P}^{(n)}$. From the assumption it follows that $\bar{P}^{(n)} \neq 0$. We shall obtain from the artinian property of of superalgebra $\bar{L}$ that $\bar{P}^{(n)}$ contains the minimal ideal $\bar{\rho}$.

Then either $[\bar{\rho}, \bar{P}] = 0$, or the ideal $\bar{\rho}$ is irreducible as a module. In the latter case the algebra $\bar{L}$ generates the primitive graded associative superalgebra B in a ring $End(\bar{\rho})$, which is a homomorphic image of superalgebra $\mathrm{Ad}(L)$. According to the graded analogue of the theorem of Kaplansky [17] the algebra B is central simple, finite dimensional over the graded center Z, which dimension is not higher than $[d/2]^2$. Hence, the dimension of algebra $\bar{P}$ over Z is not higher than n. An ideal $\bar{P}$ is locally soluble. Then the image of an ideal $\bar{P}^{(n)}$ in the ring of endomorphisms $End(\bar{\rho})$ is equal to zero. Hence, $[\bar{\rho}, \bar{P}^{(n)}] = 0$.

Passing to inverse images in algebra L we shall obtain $[\rho, P^{(n)}] \subseteq W$. Hence, $[[\rho, P^{(n)}], P^{(n)}] = 0$. Then $[\rho, P^{(n+1)}] = 0$. From the equality $P^{(n)} = P^{(n+1)}$ the equality $[\rho, P^{(n)}] = 0$ follows. Hence $\rho \subseteq W$, that contradicts the definition of ρ. The obtained contradiction proves the theorem.

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