BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(43), 2003, Pages 83–91
ISSN 1024–7696

Approximate solution of the Dirichlet problem in a circle

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Abstract. The approaches to the solution of Dirichlet problem in a unit radius circle are constructed in the manner of rational functions. There were found the estimates of approaches' inaccuracies. Assuming that the boundary condition is to be a measurable bounded function with the finite number of discontinuities. Constructions use the solution of trigonometric problem of moments.

Mathematics subject classification: 30C80.
Keywords and phrases: Dirichlet problem, rational functions, trigonometric problem of moments, orthonormalized polynomials, rational approximation.

1 Introduction

Consider the Dirichlet problem in the unit circle Ω:

$$\Delta U = 0, \qquad (x,y) \in \Omega, \tag{1}$$
$$U(\cos\theta, \sin\theta) = \varphi(\theta), \qquad \theta \in [0, 2\pi]. \tag{2}$$

The problem (1), (2) can be resolved in a polar coordinate system using the variables separation method. In this case the solution is written in the following way:

$$u(r,\phi) = C + \sum_{n=1}^{\infty} r^n \big(A_n \cos(n\phi) + B_n \sin(n\phi)\big), \tag{3}$$

where the coefficients C, $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ are calculated according to well-known formulas. The right-side formula (3) is summed up to the Poisson integral. In some cases the Poisson integral is simply calculated with the help of the residue theory. It happens, for example [1], if the function

$$\varphi(\phi) = \Phi(\cos\phi, \sin\phi),$$

where $\Phi\left(\dfrac{z+z^{-1}}{2}, \dfrac{z-z^{-1}}{2i}\right)$ is a function, regular in the Γ points and meromorphic in the Ω circle, and having there only a finite number of poles. In general cases we make use of approximate solution methods. The choice of various approximate problem solution methods (1), (2) depends on a boundary condition smoothness $\varphi(\phi)$. It is assumed in the given work, that the real valued function $\varphi(\phi)$ is measurable, bounded and having a finite number of discontinuities. In such situation polynomial approximations on the basis of formula (3) are not valid. For getting some rational approximations to find the problem's solution (1), (2) we should use the following construction. Any real valued, measurable on the closed interval $[0, 2\pi]$ and bounded function $\psi(\phi)$ can be represented as the following:

$$\psi(\phi) = \psi_+(\phi) - \psi_-(\phi), \qquad \phi \in [0, 2\pi], \tag{4}$$

where

$$\psi_+(\phi) = \{\psi(\phi),\ \text{if } \psi(\phi) \geq 0 \text{ and } 0,\ \text{if } \psi(\phi) < 0\},$$
$$\psi_-(\phi) = \{0,\ \text{if } \psi(\phi) \geq 0 \text{ and } -\psi(\phi),\ \text{if } \psi(\phi) < 0\}.$$

Moreover, if $\psi(\phi)$ is a function of bounded variation, then it can be represented in the form

$$\psi(\phi) = \int_0^\theta |\psi'(\xi)|\, d\xi - \left(\int_0^\theta |\psi'(\xi)|\, d\xi - \psi(\phi)\right), \tag{5}$$

that is in the form of two monotone non-decreasing functions.

It follows from the formula (4) (or (5)) that functions $\psi_+(\phi)$ and $\psi_-(\phi)$ are nonnegative, measurable, bounded and with a finite number of discontinuities.

Consider two auxiliary problems:

$$\Delta U_+ = 0, \qquad z \in \Omega, \tag{6}$$
$$U_+(e^{i\phi}) = \varphi_+(\phi), \qquad \phi \in [0, 2\pi] \tag{7}$$

and

$$\Delta U_- = 0, \qquad z \in \Omega, \tag{8}$$
$$U_-(e^{i\phi}) = \varphi_-(\phi), \qquad \phi \in [0, 2\pi]. \tag{9}$$

It is known, that the problem (1), (2) has a unique solution concerning the function φ with the help of made suggestions. So the problem (6), (7) and the problem (8), (9) also has a unique solution. Therefore,

$$U = U_+ - U_-.$$

That is enough to consider the case of $\varphi(\phi) \geq 0$ with almost all $\phi \in [0, 2\pi]$. Further it is supposed unconditionally, that a boundary condition is a nonnegative function. In conclusion of the item we should remember the fact, that the suggestion of boundary condition's insufficiency cannot be omitted, as the theorem of the problem's unique solution (1), (2) would be false. The following function can serve as an example:

$$V(x, y) = \frac{1 - x^2 - y^2}{(x-1)^2 + y^2}.$$

This function satisfies Laplace equation in the radius 1 circle. It is continuous up to the circle's boundary except for point (1.0). The function is identically equal to zero and it also satisfies all these conditions.

2 Rational approximations' building process

We write the solution to the problem (1), (2) according to Shwarz formula:

$$U(x,y) = Re\left[\frac{1}{2\pi i}\oint\limits_{|\xi|=1}\varphi(\xi)\frac{(\xi+z)}{(\xi-z)}\frac{d\xi}{\xi}\right],$$

where $z = x + iy$ or

$$U(x,y) = Re\left[\frac{1}{2\pi i}\oint\limits_{0}^{2\pi} U\left(e^{i\theta}\right)\frac{(e^{i\theta}+z)}{(e^{i\theta}-z)}\,d\theta\right].$$

This formula will be used below.

Denote:

$$\sigma(\phi) = \frac{1}{\int\limits_0^{2\pi}\varphi(\xi)\,d\xi}\int\limits_0^{\phi}\varphi(\xi)\,d\xi.$$

For an easier narration we suggest, that the function σ is continuous. Then the following formula is true:

$$\int\limits_0^{2\pi}\chi(\phi)\,d\sigma(\phi) = \int\limits_0^{2\pi}\chi(\phi)\varphi(\phi)\,d\phi,$$

where $\chi(\phi)$ is a continuous function, and there is Riehmann common integral at the right side of the formula.

Denote $\{P_n(z)\}_{n=0}^{\infty}$ as the sequence of orthonormalized polynomials relative to the positive measure $d\sigma$ on a radius 1 circle. These polynomials satisfy the following conditions:

$$\frac{1}{2\pi}\int\limits_0^{2\pi}P_n(e^{i\phi})\overline{P_m(e^{i\phi})}\,d\sigma(\phi) = \begin{cases}0, & n\neq m,\\ 1, & n=m.\end{cases}$$

For making such polynomials we can use the Hilbert-Schmidt orthogonalization process for a sequence $\{e^{in\phi}\}_0^{\infty}$ in the Hilbert space $L_2^{\sigma}(0,2\pi)$ with the inner product:

$$\langle f,g\rangle = \frac{1}{2\pi}\int\limits_0^{2\pi}f(\phi)\overline{g(\phi)}\,d\sigma(\phi).$$

It is known, that a sequence of orthonormalized polynomials can be built using the following sequence of moments $d\sigma$:

$$C_k = \frac{1}{2\pi}\int\limits_0^{2\pi}e^{-ik\theta}\,d\sigma(\phi), \qquad (\pm k = 0,1,2,\ldots).$$

The solution of the $d\sigma$ measure finding problem with a given set of numbers C_k is called the solution of moments' trigonometric problem and it comes essentially towards the spectral theory of the second order's finite differences equation (P.L. Chebyshev (1858), A.A. Markov (1884), T.J. Stieltjes (1894), H.L. Hamburger (1920) and others). Extensive reading materials are devoted to the discussion of the problem field (see, for example, [2, 3]).

Let $\{\alpha_n\}_{n=0}^{\infty}$ be the coefficients at orthogonal polynomials' highest degrees $\{P_n(z)\}_{n=0}^{\infty}$. The numbers

$$a_n = -\frac{\overline{P_{n+1}(0)}}{\overline{\alpha_{n+1}}}, \qquad (n = 0, 1, 2, \ldots)$$

play an important role in the theory of orthogonal polynomials in a circle. They are called circular parameters.

Consider the second order differences equation.

$$a_n y_{n+2} - (a_n + a_{n+1} z) y_{n+1} + a_{n+1} z (1 - |a_n|^2) y_n = 0. \tag{10}$$

We join some boundary conditions to the equation (10):

$$y_0 = 1, \; y_1 = 1 + a_0 z \tag{11}$$

or boundary conditions:

$$y_0 = 1, \; y_1 = 1 - a_0 z. \tag{12}$$

We denote the problem's solution (10), (11) as $\overset{*}{\psi}_n(z)$, and the problem's solution (10), (12) as $\overset{*}{\Phi}_n(z)$.

Suppose

$$\nu = \int_0^{2\pi} \varphi(\xi)\, d\xi.$$

It is said, that the Stieltjes positive measure $d\sigma$ on closed interval $[0, 2\pi]$, having an integrable density $\varphi(\phi)$ at the Lebesgue measure, satisfies the condition of Szegö, if

$$\int_0^{2\pi} \ln\left(\frac{\varphi(\phi)}{\nu}\right) d\phi > -\infty.$$

We will use standard designations below:

$$\ln^+(x) = \{\ln(x), \text{ if } x \geq 1 \text{ and } 0, \text{ if } 0 < x < 1\},$$
$$\ln^-(x) = \{0, \text{ if } x \geq 1 \text{ and } -\ln(x), \text{ if } 0 < x < 1\}.$$

Theorem 1. *Assume, that the $d\sigma$ measure satisfies the condition of Szegö, then uniformly on the compacts $|z| \le \nu < 1$*

$$\lim_{n\to\infty} \frac{\nu}{\left|\overset{*}{\Phi}_n(z)\right|^2 \alpha_n^2} = \mathrm{Re}(\hat{U}(z)),$$

where

$$\hat{U}(z) = \frac{1}{2\pi i} \oint\limits_{|\xi|=1} \varphi(\xi) \frac{(\xi+z)}{(\xi-z)} \frac{d\xi}{\xi} + i\mathrm{Im}\hat{U}(0).$$

Proof. A sequence of polynomials $\overset{*}{\Phi}_n(z)$ converges uniformly on the compact subsets of the unite circle (see, for example, [2, p. 141]) towards the function $\dfrac{D(0)}{D(z)}$, where

$$D(z) = \exp\left(\frac{1}{4\pi}\int\limits_0^{2\pi} \ln\left(\frac{\varphi(\phi)}{\nu}\right) \frac{(e^{i\theta}+z)}{(e^{i\theta}-z)} d\phi\right).$$

Moreover,

$$\ln\left(\frac{\varphi(\phi)}{\nu}\right) \in L_1(0, 2\pi),$$

and the function

$$D(z) \in H^2,$$

where H^2 is Hardy space, and is the only thing of the space to satisfy the equality

$$\left|D\left(e^{i\theta}\right)\right|^2 = \frac{\varphi(\phi)}{\nu}. \tag{13}$$

According to the logarithm definition we get

$$2\ln\big(D(z)\big) = \frac{1}{2\pi}\int\limits_0^{2\pi} \ln\left(\frac{\varphi(\phi)}{\nu}\right) \frac{(e^{i\theta}+z)}{(e^{i\theta}-z)} d\phi,$$

where the logarithm branch is allocated with a condition $\mathrm{Im}(\ln(D(0))) = \phi$.

Consider the function

$$\nu e^{2\ln(D(z))} = \nu D^2(z).$$

The equality is true for

$$\mathrm{Re}\left(\nu e^{2\ln(D(z))}\right) = \mathrm{Re}\left(\nu D^2(z)\right) = \nu \mathrm{Re}\left(D^2(z)\right) = \nu |D(z)|^2. \tag{14}$$

It follows from the formulas (13), (14), that the unit circle harmonic function $\mathrm{Re}\left(\nu D^2(z)\right)$ takes a meaning which is equal to $\varphi(\phi)$ at the circle's boundary, and therefore has the sought solution of Dirichlet problem (1), (2). In particular,

$$\hat{U}(z) = \nu e^{2\ln(D(z))} = \nu D^2(z). \tag{15}$$

Note, that the function $D(z)$ does not have zeros inside the unit circle and

$$\sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} \ln^+\left(\left|D\left(re^{i\phi}\right)\right|\right) d\varphi < +\infty.$$

Furthermore, the following equality ($|z|<1$) is true

$$\frac{1}{\left|\overset{*}{\Phi}_n(z)\right|^2} \le \frac{\alpha_n^2}{\alpha_0^2(1-|z|^2)}$$

and it is uniform at $|z| \le r < 1$ (see [3], p. 14 and p. 26)

$$\lim_{n\to\infty} \frac{\nu}{\left|\overset{*}{\Phi}_n(z)\right|^2 \alpha_n^2} = \mathrm{Re}(\hat{U}(z)). \tag{16}$$

Equality (16) shows, that the sequence of rational functions

$$\frac{\nu}{\left|\overset{*}{\Phi}_n(z)\right|^2 \alpha_n^2}$$

is an approximation to the solution of Dirichlet problem (1), (2). Notice that we consider the case of nonnegative boundary condition. The theorem has been proved.

We find the convergence speed rating of rational approximations shown in the theorem 1. Denote:

$$\delta_n = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\chi_E(\phi)}{D\left(e^{i\phi}\right)} - \alpha_n^2\, \overset{*}{\Phi}_n\!\left(e^{i\phi}\right) D(0)\right|^2 d\sigma(\phi)\right)^{\frac{1}{2}},$$

where $\chi_E(\phi)$ is the characteristic function of the set of points E, where exists a finite and positive derivative $\dfrac{d\sigma(\phi)}{d\phi}$. It is known, that almost everywhere $\dfrac{d\sigma(\phi)}{d\phi} = \varphi(\phi)$. If for measure the condition of Szegö is met, then $\delta_n \to 0$ with $n\to\infty$. In the next theorem the condition of Szegö is supposed to be met.

Theorem 2. *The inequalities are true for all z, $|z|<1$:*

$$\left|\frac{\nu}{\left|\overset{*}{\Phi}_n(z)\right|^2 \alpha_n^2} - \mathrm{Re}\left[\hat{U}(z)\right]\right| \le \frac{D(0)^2\nu\delta_n}{(1-|z|^2)\,\alpha_0^4}\left(\frac{1}{\sqrt{1-|z|}} + \frac{D(0)^2\delta_n}{D(0)^2+\alpha_0}\right)\times$$

$$\times\left(2\alpha_0 + D(0)^2\delta_n\left(\frac{1}{\sqrt{1-|z|}} + \frac{D(0)^2\delta_n}{D(0)^2+\alpha_0}\right)\right).$$

Proof. For points z lying inside the unit circle we have:

$$\nu D^2(z) - \frac{\nu}{\left(\overset{*}{\Phi}_n(z)\right)^2 \alpha_n^2} = \frac{\nu}{\left(\overset{*}{\Phi}_n(z)\right)^2}\left(\alpha_n^2\left(\overset{*}{\Phi}_n(z)\right)^2 D^2(z) - 1\right) =$$
$$= \frac{\nu}{\left(\overset{*}{\Phi}_n(z)\right)^2}\left(2\left(\alpha_n \overset{*}{\Phi}_n(z)D(z) - 1\right) + \left(\alpha_n \overset{*}{\Phi}_n(z)D(z) - 1\right)^2\right). \tag{17}$$

From the formula (17) we get:

$$\left|\left|\nu D^2(z)\right| - \left|\frac{\nu}{\overset{*}{\Phi}_n(z)^2\alpha_n^2}\right|\right| \le \left|\nu D^2(z) - \frac{\nu}{\overset{*}{\Phi}_n(z)^2\alpha_n^2}\right| \le$$
$$\le \frac{\nu}{\alpha_n^2\left|\overset{*}{\Phi}_n(z)^2\right|}\left(2\left|\alpha_n \overset{*}{\Phi}_n(z)D(z) - 1\right| + \left|\alpha_n \overset{*}{\Phi}_n(z)D(z) - 1\right|^2\right). \tag{18}$$

Estimate the right side of the formula (18). First of all we note, that

$$\frac{\nu}{\alpha_n^2\left|\overset{*}{\Phi}_n(z)^2\right|} = \frac{\nu}{\alpha_n^2\left|\overset{*}{\Phi}_n(z)\right|^2} \le \frac{\nu}{\alpha_0\left(1 - |z|^2\right)}. \tag{19}$$

Secondly, this estimation is true (see [3, p. 108–109]):

$$\left|\alpha_n \overset{*}{\Phi}_n(z)D(z) - 1\right| \le \frac{\delta_n D(0)^2}{\alpha_0}\left(\frac{1}{\sqrt{1-|z|}} + \frac{\delta_n D(0)^2}{D(0)^2 + \alpha_0}\right). \tag{20}$$

It follows from (18) – (20):

$$\left|\frac{\nu}{\left|\overset{*}{\Phi}_n(z)\right|^2\alpha_n^2} - \operatorname{Re}\left[\hat{U}(z)\right]\right| = \left|\left|\nu D^2(z)\right| - \left|\frac{\nu}{\overset{*}{\Phi}_n(z)^2\alpha_n^2}\right|\right| \le$$
$$\le \frac{\nu}{\alpha_0^2\left(1-|z|^2\right)}\left(\frac{2\delta_n D(0)^2}{\alpha_0}\left(\frac{1}{\sqrt{1-|z|}} + \frac{\delta_n D(0)^2}{D(0)^2+\alpha_0}\right) + \right.$$
$$\left. + \frac{\delta_n^2 D(0)^4}{\alpha_0^2}\left(\frac{1}{\sqrt{1-|z|}} + \frac{\delta_n D(0)^2}{D(0)^2+\alpha_0}\right)^2\right). \tag{21}$$

Transforming the right side of formula (21) we get the sought estimation:

$$\left|\frac{\nu}{\left|\overset{*}{\Phi}_n(z)\right|^2\alpha_n^2} - \operatorname{Re}\left[\hat{U}(z)\right]\right| \le \frac{D(0)^2\nu\delta_n}{\left(1-|z|^2\right)\alpha_0^4}\left(\frac{1}{\sqrt{1-|z|}} + \frac{D(0)^2\delta_n}{D(0)^2+\alpha_0}\right) \times$$

$$\times\left(2\alpha_0 + D(0)^2\delta_n\left(\frac{1}{\sqrt{1-|z|}} + \frac{D(0)^2\delta_n}{D(0)^2+\alpha_0}\right)\right).$$

The theorem has been proved.

The rate at which sequence δ_n decreases with $n \to \infty$ depends on the function $\varphi(\phi)$ properties (see [3, p. 199, table I]).

Consider another more general case, when the function $\varphi(\phi)$ cannot satisfy the term of Szegö.

Theorem 3. *Uniformly on the compacts* $|z| \leq r < 1$

$$\hat{U}(z) = C_0\nu \lim_{n\to\infty} \frac{\overset{*}{\Psi}_n(z)}{\overset{*}{\Phi}_n(z)}$$

and for the solution of Dirichlet problem $\left(\mathrm{Re}\left(\hat{U}(z)\right)\right)$ *the following estimation of approximations convergence rate is true:*

$$\left|\mathrm{Re}\left(\hat{U}(z)\right) - \frac{\nu}{2\pi}\mathrm{Re}\left(\frac{\overset{*}{\Psi}_n(z)}{\overset{*}{\Phi}_n(z)}\right)\right| \leq \frac{\sqrt{2}r^n\nu}{2\pi(1-r)^{\frac{3}{2}}}.$$

Proof. It follows directly from [3, p. 16 and p. 160]. The theorem has been proved. We pay attention to the circumstance, that the equalities:

$$\frac{1}{2\pi}\mathrm{Re}\left(\frac{\overset{*}{\Psi}_n(z)}{\overset{*}{\Phi}_n(z)}\right) = \frac{1}{\left|\overset{*}{\Phi}_n(z)\right|^2 \alpha_n^2}$$

are true, in general, at the unit circle boundary only [3, p. 17].

3 Final remarks

So, the rational functions sequence $R_n(z) = \mathrm{Re}\left(\frac{\overset{*}{\Psi}_n(z)}{\overset{*}{\Phi}_n(z)}\right)$ (in case of the condition of Szegö fulfillment $R_n(z) = \frac{\nu}{\left|\overset{*}{\Phi}_n(z)\right|^2 \alpha_n^2}$) converges uniformly inside the unit circle to the solution of Dirichlet problem (1), (2) with nonnegative boundary condition φ. In general case we denote the sequence of rational approximations for the problem (6), (7) via $R_n^+(z)$, and for the problem (8), (9) via $R_n^-(z)$. The sequence of rational functions $R_n^+(z) = R_n^+(z) - R_n^-(z)$ will be converging uniformly on the compacts inside the unit circle to the solution of the problem (1), (2). The approximations ratings can be made from the above-proved theorems.

The calculations of polynomials $\overset{*}{\Psi}_n(z)$ and $\overset{*}{\Phi}_n(z)$ are achieved easily. For this we should use the recurrent equation (10) and notice, that circular parameters can be found according to formulas:

$$a_n = \frac{(-1)^n}{\overline{\Delta}_{n-1}} \begin{vmatrix} \overline{C}_{-1} & \overline{C}_{-2} & \dots, & \overline{C}_{-n} \\ \overline{C}_{-0} & \overline{C}_{-1} & \dots, & \overline{C}_{-n+1} \\ \vdots & \vdots & & \vdots \\ \overline{C}_{n-2} & \overline{C}_{n-3} & \dots, & \overline{C}_{-1} \end{vmatrix},$$

where Δ_k is Toeplitz matrix determinants

$$\{C_{p-q}\}_{p,q=0}^{k}. \tag{22}$$

There exist some special methods for matrix determinant calculation in the view of (22).

The author has realized the algorithms of rational approximations construction within the computer technical system *Mathematica*.

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Received November 12, 2003