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X-normal mappings

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Abstract. This is a survey of achievements in the theory of normal holomorphic mappings. We systematize and present all the results on the subject that are obtained by the author from the beginning of the theory until the date of writing the paper.

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1 Introduction

The idea to connect with a meromorphic function f in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ a family $\mathcal{F} = \{f \circ g\}$, where g ranges over all automorphisms of U (one-to-one holomorphic mappings of U onto itself) and study those functions f whose family $\mathcal{F}$ is normal was arise apparently of K. Yosida [36] in 1934 and considered by K. Noshiro [27] in 1937. O. Lehto and K.I. Virtanen [26] call "normal functions" those meromorphic functions f whose family $\mathcal{F}$ is normal.

The results obtained by O. Lehto and K.I. Virtanen [26] in 1957 motivated further study of normal meromorphic function. In the period between 1957 and 1965, a significant contribution to the theory was madden by F. Bagemihl, W. Seidel, V.I. Gavrilov, P. Lappan.

A systematic study of normal functions in $\mathbb{C}^n$ was begun by the author in 1981 in the papers [9–11]. In 1983 the first dissertation on normal functions was defended at Moscow State University by the author and J.A. Cima and S.G. Krantz have published, in the USA, the article [6] in which they have developed the ideas contained in [8],[9]. In such a way appeared the theory of normal mappings.

The first application of this theory was obtained by V.I. Gavrilov and the author in the paper [18]. The first survey on the theory of normal mappings was published by V.I. Gavrilov and P.V. Dovbush [19] in 2001.

The fact that the new subsection *Several Complex Variables 32A18 Bloch functions, normal functions* was created in the Mathematics Subject Classification Scheme of the AMS journal Mathematical Review in 2000 emphasizes the actuality of this theme.

All results of this work belong to the author, are published in [9–20], and the author reported about them at:

International Conference on Mathematics and Informatics, Chisinau, September 19-21, 1996.

International Conference on Complex Analysis and Related Topics. The VIII$^{\text{th}}$ Romanian-Finnish Seminar. 1999, Iassy, Romania.

The first Conference of the Mathematical Society of the Republic of Moldova, Chisinau, 2001.

International Conference on Complex Analysis and Related Topics. The IX$^{\text{th}}$ Romanian-Finnish Seminar. 2001, Braşov, Romania.

The 5th Congress of Romanian mathematicians. Pitesti. Romania. June 22–28, 2003.

2 X-normal mappings

Let M and N be complex manifolds. We denote the set of all holomorphic mappings from M into N by $H(M, N)$.

A subset $\mathcal{F}$ of $H(M, N)$ is said to be a *normal* family in the sense of H.Wu [35] iff every sequence $\{f_j\}$ of $\mathcal{F}$ has a subsequence which ether converges uniformly on compact subsets of M (i.e. converges *normally* on M) or, given any compact K in M, and a compact K' in N, there exists an j_0 such that $f_j(K) \cap K' = \emptyset$ for all $j \geq j_0$.

For the complex manifolds M, which have a transitive group of biholomorphic automorphisms[1] (i.e., if given p, $q \in M$ there exists a biholomorphic automorphism $\phi : M \to M$ with $\phi(p) = q$) the definition of normal mapping can be introduced by analogy with the one dimensional case.

Definition 1. *Let M be a homogeneous manifold and N be a connected Hermitian manifold. We say that a holomorphic mapping $f : M \to N$ is normal if the family $\mathcal{F} = \{f \circ g\}$, where g ranges over all automorphisms (one-to-one holomorphic mappings of M onto itself), forms a normal family in the sense of H.Wu.*

The normality of a complex function imposes a restriction on the growth of function. Our first result is the following.

Theorem 1. *Let f be a normal meromorphic function on the unit ball $B = \{z \in \mathbb{C}^n : |z| < 1\}$ and let $\Omega = \{z \in B : |2z_n - 1| < 1, |'z| < |1 - z_n|\}$. If for all $z \in \Omega$*

$$|f(z)| < \exp\left(\frac{-1}{(1 - |z|)^{1+\varepsilon}}\right)$$

for some $\varepsilon > 0$. Then $f \equiv 0$.

It is important to note that:

> (a) The unit disc in $\mathbb{C}$ is a canonical domain, because Riemann mapping theorem says that every proper simply connected open subset D of $\mathbb{C}$ is biholomorphic to the disc. Poincaré's theorem that the ball and the polydisc are biholomorphically inequivalent, shows that there is no

[1] These manifolds are called homogeneous manifolds.

Riemann mapping theorem in several complex variables. This implies that there is no canonical domain in $\mathbb{C}^n$ for $n > 1$.

(b) It has long been known that in $\mathbb{C}^2$ there exist simply connected domains whose only holomorphic automorphism is the identity (cf. Benke, Tullen [3, p. 169]). And, what is more smoothly, bounded domains in $\mathbb{C}^n$ generally have no biholomorphic self mappings different from the identity. A result due to Burns, Shnider and Wells [5] clarifies how truly dismal the situation is.

(c) Every domain in the complex plane with C^2-boundary is strongly pseudoconvex. The result due to Bun Wong [34] and Rosay [30] states that the only strongly pseudoconvex domain in $\mathbb{C}^n$ with transitive automorphism group is the ball.

(d) E. Cartan proved that any bounded homogeneous domain in $\mathbb{C}^2$, is biholomorphic to either the ball $B^2 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 < 1\}$ or the polydisc $U^2 = \{(z_1, z_2) : |z_1| < 1, |z_2| < 1\}$. In $\mathbb{C}^3$, E. Cartan's result is that any bounded homogeneous domain is biholomorphic to either the ball, the polydisc, or (writing $z_j = x_j + y_j$) the tube domain

$$\{(z_1, z_2, z_3) : y_3 > [(y_1)^2 + (y_2)^2]^{1/2}\}.$$

(While the third of these domains is unbounded, it has a bounded realization.) In any $\mathbb{C}^n$, the set of equivalence classes of bounded symmetric domains[2] is finite, as shown by E. Cartan.

Since general domains in $\mathbb{C}^n$, $n > 1$, have trivial automorphism groups it is natural to try to generalize the notion of normal mappings to general complex manifolds and to extend classical results to more general settings.

Let N be a connected paracompact hermitian manifold with hermitian metric ds_N which induces the standard topology on N. By s_N we denote the distance function associated with ds_N. Let Y be a relatively compact complex subspace of a hermitian manifold N. We shall denote by $H(M, Y)$ the space of all holomorphic mappings $f : M \to N$ with $f(M) \subset Y$.

The classical hyperbolic metric on the unit disk can be extended to the higher dimension at least by three different ways.

Let $T_p(M)$ be a complex tangent space to M at $p \in M$ and vector $v \in T_p(M)$. The *Kobayashi norm* is given by

$$\begin{aligned} K_M(p, v) \quad = \quad & \inf\{1/r : r > 0 \text{ and there exists} \\ & h \in H(U, M), h(0) = p, h'(0) = r \cdot v\}. \end{aligned}$$

With $v \in T_p(M)$ as above, the *Caratheodory norm* is defined by

$$C_M(p, v) = \sup\{|dg_p(v))| : g \in H(M, U)\}.$$

[2]A domain $D \subseteq \mathbb{C}^n$ is called symmetric if for each $z_0 \in D$ there exists a biholomorphic automorphism $\phi_{z_0} : D \to D$ with $\phi_{z_0} \circ \phi_{z_0} = id$ so that z_0 is an isolated fixed point of ϕ_{z_0}.

The *Bergman norm*, denoted by $B_M(p,v)$, is defined by the relation[3]

$$(B_M(p,v))^2 = \sum_{j,k=1}^{n} g_{j,k}(p) v_j \overline{v}_k.$$

In following let X_M denote the Caratheodory, Kobayashi or Bergman norms on M.

First we need the following generalization of Marty's Criterion for normality of holomorphic mappings.

Lemma 1. *Let M be a complex manifold such that for each $p \in M$, there exists a neighborhood U and a constant $c = c(U) > 0$ such that $X_M(p,v) \geq c \cdot |v|$ for $(p,v) \in T(U) = U \times T_p(M)$, and let Y be a relatively compact complex subspace of an Hermitian manifold N with the Hermitian metric ds_N. The family $F \subset H(M,Y)$ is normal in the sense of H.Wu if for each compact subset $K \subset M$ there exists a positive constant $L = L(K)$ such that*

$$ds_N^2(f(p), df(p)v) \leq L(K) \cdot X_M(p,v)^2$$

for all $p \in K$, $v \in T_p(M)$ and all $f \in F$.

Marty's Criterion was first proved by the author in [8][4] for $M =$ domain in $\mathbb{C}^n$ and $N = \mathbb{C}$. See also [33][5],[6],[21].

Marty's Criterion plays a fundamental role in the theory of $\mathcal{X}$-normal mappings. Using Marty's Criterion, we can prove the following elegant geometric characterization of normal mappings.

Theorem 2. *Let M be a homogenious complex manifold such that for each $p \in M$, there exists a neighborhood U and a constant $c = c(U) > 0$ such that $X_D(p,v) \geq c \cdot |v|$ for $(p,v) \in U \times T_p(M)$ and let Y be a relatively compact complex subspace of an Hermitian manifold N with the Hermitian metric ds_N. A holomorphic mapping $f \in H(D,Y)$ is normal iff there exists a finite positive constant L such that*

$$f^* ds_N^2 \leq L \cdot (X_D)^2.$$

Results related to Theorem 2 can be found in [9, 20, 21, 22].

Using Shwarz-Pick lemma it is easy to check that if $D = U$ then $X_U(z,v)$ coincides with the Poincaré metric in U $\rho(z)|v| = (1-|z|^2)^{-1}|v|$. Hence Theorem 1 is the full generalization of the Lexto-Virtanen Criterion to a higher-dimensional domain:

A meromorphic function $f : U \to \overline{\mathbb{C}}$ is normal iff there exist a finite constant L such that

$$f^* ds^2_{\overline{\mathbb{C}}}(z,v) \leq L \cdot (\rho(z)|v|)^2,$$

for all $z \in U$, $v \in \mathbb{C}$.

The characterization of normal mappings given in Theorem 2 leads to a natural generalization of the concept of $\mathcal{X}$-normal mappings. We give the following:

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[3] $(B_M(p,v))^2$ is called Bergman's form of M.

[4] Received by the Editor on 18 July, 1979.

[5] Received by the Editor on 17 October, 1979; revised form on 30 January, 1980.

Definition 2. *Let M be a complex manifold and let N be a Hermitian manifold N with the Hermitian metric ds_N. We say that a holomorphic mapping $f : M \to N$ is $\mathcal{X}$-normal if there exists a finite positive constant L such that*

$$f^* ds_N^2 \leq L \cdot (X_M)^2.$$

One sees at once that all Bloch functions (see [33]) of several complex variables are normal functions.

It is easy to check using the definition of the Caratheodory norm that all bounded holomorphic function are C-normal. On the other hand, from the Lindelof's theorem follows that there are normal functions which do not belong to any Hardy space $H^p(D)$, and functions in H^p which are not normal.

In the case of strongly pseudoconvex domains, the classes of normal mappings defined in terms of the Bergman, Carathéodory, and Kobayashi norms are the same. This assertion follows from well-known estimates on the asymptotic behavior of these norms.

Since $X_D \geq C_D$, the class of $\mathcal{K}$-normal or $\mathcal{B}$-normal mappings contains the class of $\mathcal{C}$-normal mappings. But in what follows, we will show that these classes, generally speaking, are different.

3 Extension properties of $\mathcal{X}$-normal mappings

First we prove that in the one dimensional case the following result holds.

Proposition 1. *Let f be a meromorphic function in punctured unit disk $U^* = U \setminus \{0\}$. If f has an isolated singularity at the origin and there exists a monotone increasing function h such that*

$$|zf'(z)| \leq h(|f(z)|)$$

for all $z \in U^*$, then f has a meromorphic extension at the origin.

If $D \subset \mathbb{C}$ is multiply connected, f is said to be normal on D if f is normal on the universal cover of D.

From Proposition 1 immediately follows the extension of the big Picard theorem due to O. Lehto and K.I. Virtanen [26, Theorem 9, p. 92]:

> *Isolated singularities are removable for normal meromorphic functions of the one complex variable.*

Since norms K_U and K_{U^*} are comparable near ∂U, the extended function is normal in U.

It is of interest to have analogues of this theorem in several variables.

Generalizing O. Lehto and K.I. Virtanen's result to the case of several complex variables P.Järvi [24] proved that $\mathcal{K}$-normal mappings can be extended to holomorphic mappings through analytic subvarieties of codimension 1 provided the singularities are normal crossings.

J. Riihentaus [29] generalized P.Järvy result and proved that $\mathcal{K}$-normal mappings can be extended to holomorphic mappings through closed in D subsets of locally finite $(2n-2)$-dimensional Hausdorff measure.

The principal our result is the following:

Theorem 3. *Suppose that D is a bounded domain in $\mathbb{C}^n$, $n > 1$, such that X_D is a continuous function on $D \times \mathbb{C}^n$, and suppose that $E \subset D$ is closed in D and has the zero $(2n-1)$-dimensional Hausdorff measure and such that $X_{D\setminus E} \equiv X_D$ on $(D \setminus E) \times \mathbb{C}^n$. If $f : D \setminus E \to \overline{\mathbb{C}}$ is $\mathcal{X}$-normal, then f extends to a holomorphic mapping $F : D \to \overline{\mathbb{C}}$ which is $\mathcal{X}$-normal on D.*

If $E \subset D$ is closed in D and has the zero $(2n-1)$-dimensional Hausdorff measure then $C_{D\setminus A}(z,v) \equiv C_D(z,v)$ on $(D \setminus A) \times \mathbb{C}^n$. Since Caratheodory norm $C_D(z,v)$ is a continuous function on $D \times \mathbb{C}^n$ as a consequence of Theorem 3 we have

Theorem 4. *Let $D \subset \mathbb{C}^n$, $n > 1$, be a domain and let $E \subset D$ be closed in D and have the zero $(2n-1)$-dimensional Hausdorff measure. If $f : D\setminus A \to \overline{\mathbb{C}}$ is $\mathcal{C}$-normal mapping, then f has a $\mathcal{C}$-normal extension $F : D \to \overline{\mathbb{C}}$.*

If A is an analytic subset of D of codimension at least one, then $B_{D\setminus A}(z,v) \equiv B_D(z,v)$ on $(D \setminus A) \times \mathbb{C}^n$ (see [4]). Bergman norm $B_D(z,v)$ is a continuous function on $D \times \mathbb{C}^n$. It follows that Theorem 3 has the following consequence:

Theorem 5. *Let $D \subset \mathbb{C}^n$, $n > 1$, be a domain and let $A \subset D$ be an analytic subvariety of codimension at least one. If $f : D \setminus A \to \overline{\mathbb{C}}$ is $\mathcal{B}$-normal mapping, then f has a $\mathcal{B}$-normal extension $F : D \to \overline{\mathbb{C}}$.*

Since we can consider any $a \in U$ as an analytic subset of U of codimension one we can interpret Theorem 5 as the full generalization of classical Lehto-Virtanen's theorem [26, Theorem 9, p. 92] to a higher-dimensional domain.

Using the notion of $\mathcal{P}$-sequence (a sequence $\{z_j\} \subset D$ is a $\mathcal{P}$-sequence for an holomorphic mapping $f : D \to \overline{\mathbb{C}}$ if $\lim_{j\to\infty} k_D(z_j, w_j) = 0$ but $\overline{\lim}_{j\to\infty} s_{\overline{\mathbb{C}}}(f(z_j), f(w_j)) \geq \epsilon$ for some $\epsilon > 0$ and some $\{w_j\} \subset D$) we prove the following result.

Theorem 6. *Let $D \subset \mathbb{C}^n$, $n > 1$, be a domain and let $E \subset D$ be closed in D and have the zero $(2n-2)$-dimensional Hausdorff measure. If $f : D\setminus A \to \overline{\mathbb{C}}$ is $\mathcal{K}$-normal mapping, then f has a $\mathcal{K}$-normal extension $F : D \to \overline{\mathbb{C}}$.*

It is shown in [22] that all mappings whose range omits at least three values belong to the class of all K-normal mappings. The following simple example shows that this is not true for the rest two classes mentioned above.

Let $f(z_1, z_2) = z_1/z_2$ and let $A = \{z_1z_2(z_1 - z_2) = 0\}$. Because $f(D \setminus A) \subset \mathbb{C} \setminus \{0, 1\}$, it follows that f is $\mathcal{K}$-normal in $D \setminus A$. The function f can not be $\mathcal{C}$-normal or $\mathcal{B}$-normal in $D \setminus A$. Otherwise f would have a holomorphic extension $F : D \to \overline{\mathbb{C}}$.

Therefore, we have the following proposition.

Proposition 2. *Let D be a domain in $\mathbb{C}^2$ and let $A = \{z_1z_2(z_1 - z_2) = 0\}$.*

(a) The class of $\mathcal{C}$-normal mappings defined on $D \setminus A$ is a proper subclass of $\mathcal{K}$-normal mappings defined on $D \setminus A$.

(b) The class of $\mathcal{B}$-normal mappings defined on $D \setminus A$ is different from the class of $\mathcal{K}$-normal mappings defined on $D \setminus A$.

(c) The Kobayashi norm on $D \setminus A$ is not compatible with the Bergman or the Caratheodory norm on $D \setminus A$.

4 Boundary behavior of holomorphic mappings

In the case of one complex variable, O. Lehto and K.I. Virtanen [26] showed that the notion of normal meromorphic functions is closely related to some important problems from the theory of boundary behavior of holomorphic mappings.

Let D be a bounded domain in $\mathbb{C}^n$ with C^2-boundary and let $\delta(z) = \inf\{|z-\zeta| : \zeta \in \partial D\}$. If $\xi \in \partial D$, let ν_ξ denote the unit outward normal at ξ.

We say that $f \in H(D,N)$ has *radial limit* $l \in N$ at $\xi \in \partial D$ if

$$\lim_{t \to 0^+} s_N(f(\xi - t\nu_\xi), l) = 0.$$

An admissible approach region $\mathcal{A}_\alpha(\xi)$ with the vertex at $\xi \in \partial D$ and of the aperture $\alpha > 0$ is defined as follows ([32]):

$$\mathcal{A}_\alpha(\xi) = \{z \in D \ : \ |(z-\xi, \nu_\xi)| < (1+\alpha)\delta_\xi(z), |z-\xi|^2 < \alpha\delta_\xi(z)\},$$

where $(\, , \,)$ is the usual Hermitian product in $\mathbb{C}^n$, and $\delta_\xi(z) = \min\{\delta(z), dist(z, T_\xi(\partial D))\}$.

We say that $f \in H(D,N)$ has an *admissible limit* $l \in N$ at $\xi \in \partial D$ if

$$\lim_{\mathcal{A}_\alpha(\xi) \ni z \to \xi} s_N(f(z), l) = 0,$$

for every $\alpha \geq 1$.

E. Stein [32] proves that admissible domains give a Fatou-type theorem on any smoothly bounded domain in $\mathbb{C}^n$, but his result is only optimal for strongly pseudoconvex domains (see [23]). In Stein's theory the aperture α of the approach regions is fixed once and for all.

We prove the following theorems.

Theorem 7. *Let D be a bounded domain in $\mathbb{C}^n$, $n > 1$, with C^2-boundary. If $f \in H(D,\mathbb{C})$ has the radial limit $l \in \mathbb{C}$ at $\xi \in \partial D$ and Ref has admissible limit at ξ, then f has an admissible limit at ξ.*

Theorem 8. *Let D be a bounded domain in $\mathbb{C}^n$ with C^2-boundary. If $f : D \to \overline{\mathbb{C}}$ is $\mathcal{K}$-normal in D, and*

$$\lim_{\mathcal{A}_\beta(\xi) \ni z \to \xi} s_{\overline{\mathbb{C}}}(f(z), l) = 0 \text{ exists for some } \beta > 0,$$

then f has an admissible limit $l \in \overline{\mathbb{C}}$ at ξ.

If $\alpha \geq 0$, define the *$\mathcal{K}$-admissible approach region* of aperture α at ξ to be (see [25])

$$\mathcal{K}_\alpha(\xi) = \{z \in D \ : \ k_D(z, N_\xi) < \alpha\}.$$

Here $k_D(z, N_\xi)$ represents the Kobayashi distance from z to N_ξ.

If $D \subset\subset \mathbb{C}^n$ is strongly pseudoconvex domain then there are constants $c_1, c_2 > 0$ depending on D and an open set $W \supseteq \partial D$ such that

$$U \cap \mathcal{A}_{c_1\alpha}(\xi) \supseteq \mathcal{K}_\alpha(\xi) \cap W \supseteq U \cap \mathcal{A}_{c_2\alpha}(\xi).$$

for any $\xi \in \partial D$ and $\alpha > 1$.

We say that a mapping $g : D \to N$ has the *$\mathcal{K}$-limit* $l \in N$ at $\xi \in \partial D$ if

$$\lim_{\mathcal{K}_\alpha(\xi) \ni z \to \xi} s_N(g(z), l) = 0,$$

for every $\alpha \geq 1$.

Denote by

$$Q_f(z) = \sup_{v \in \ \mathbb{C}^n \setminus \{0\}} \left\{ \frac{ds_N(f(z), df(z)(v))}{K_D(z, v)} \right\}.$$

In [26], O. Lehto and K.I. Virtanen showed that if a meromorphic function f in the unit disk U has the radial limit at the point $\mathbf{1} \in \partial U$, then f has the angular limit at $\mathbf{1}$ iff Q_f is bounded on every Stolz regions at $\mathbf{1}$.

This is not longer true for several variables. The function $f(z_1, z_2) = z_2^{2m}/(1-z_1)$ is bounded and holomorphic on the Tullen domain $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$ and f has the radial limit 0 at $\mathbf{1} = (1, 0)$ but it does not have a $\mathcal{K}$-limit at $\mathbf{1}$.

We prove the following criterion for existence of $\mathcal{K}$-limits.

Theorem 9. *Let D be a complete hyperbolic domain in $\mathbb{C}^n$ and let Y be a relatively compact complex subspace of an Hermitian manifold N with the Hermitian metric ds_N. If $f \in H(D, Y)$ has the radial limit at $\xi \in \partial D$, then f has the $\mathcal{K}$-limit at ξ iff Q_f has the $\mathcal{K}$-limit zero at ξ.*

In [2], F. Bagemihl and W. Seidel posed the following question:

Given a sequence $\{z_j\} \subset U$ converging to same $\xi \in \partial U$ and a holomorphic mapping $f \in H(U, \overline{\mathbb{C}})$ such that $\lim_{j\to\infty} s_{\overline{\mathbb{C}}}(f(z), l) = 0$ for same $l \in \overline{\mathbb{C}}$, under what condition on f and $\{z_j\}$ can f have the limit l along some continuum in U which is asymptotic at ξ.

They answer this question with two interesting sufficient condition on f and $\{z_j\}$. We extend their results to the higher dimensional case.

Theorem 10. *Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$ $(n \geq 1)$, $\xi \in \partial D$, and let Y be a relatively compact complex space of an Hermitian manifold N with the Hermitian metric ds_N. Let $f \in H(D, Y)$ be a normal mapping which omits $l \in \overline{Y}$ in D. Let $\{a^m\}$ and $\{b^m\}$ be sequences in D such that*

$$\lim_{m\to\infty} a^m = \xi \in \partial D \quad \text{and} \quad \lim_{m\to\infty} b^m = \xi.$$

If $k_D(a^m, b^m) < \epsilon < \infty$ *for all* $m \geq 1$ *and*

$$\lim_{m\to\infty} s_N(f(a^m), l) = 0, \quad \textit{then} \quad \lim_{m\to\infty} s_N(f(b^m), l) = 0.$$

The same results holds when we replace "strongly pseudoconvex" by "convex".

From Theorem 10 immediately follows the following strengthening of the Lindelöf-Lehto-Virtanen's theorem:

Theorem 11. *Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$ $(n \geq 1)$, $\xi \in \partial D$, and let Y be a relatively compact complex space of an Hermitian manifold N with the Hermitian metric ds_N. If $f \in H(D,N)$ is a normal mapping which omits $l \in \overline{Y}$ in D and f has radial limit l at ξ, then f has the admissible limit at ξ.*

The hypothesis of "radial limit" may be replaced by "limit along some non-tangential curve γ" as proved by the author in [11][6]. In [28][7] this theorem was proved for $D = B$ and $f \in H^\infty(B)$.

Theorem 10 also holds when we replace "strongly pseudoconvex" by "convex".

The following theorem illustrates more precisely the Lindelöf principle:

Theorem 12. *Let D be a convex domain in $\mathbb{C}^n$ $(n \geq 1)$, $\xi \in \partial D$, and let Y be a relatively compact complex space of an Hermitian manifold N with the Hermitian metric ds_N. If $f \in H(D,N)$ is a normal mapping which omits $l \in \overline{Y}$ in D and f has radial limit l at ξ, then f has the $\mathcal{K}$-limits at ξ.*

Again the hypothesis of "radial limit" may be replaced by "limit along some non-tangential curve γ."

It should be noted that, in general, $\mathcal{K}$-admissible domains are **strongly larger** than admissible domains.

A hypoadmissible approach region $\mathcal{A}_\alpha^\epsilon(\xi)$, $0 < \epsilon < 1$, with vertex $\xi \in \partial D$ and aperture $\alpha > 0$ is defined as follows ([7]):

$$\mathcal{A}_\alpha^\epsilon(\xi) = \{\, z \in D \ : \ |(z-\xi, \nu_\xi)| < (1+\alpha)\delta_\xi(z), |z-\xi|^2 < \alpha\delta_\xi^{1+\epsilon}(z)\,\}.$$

We say that a mapping $f \in H(D,Y)$ has the *hypoadmissible limit* $l \in \overline{Y}$ at $\xi \in \partial D$, if for every $\alpha > 0$ and $\epsilon, 0 < \epsilon < 1$,

$$\lim_{\mathcal{A}_\alpha^\epsilon(\xi) \ni z \to \xi} s_N(f(z), l) = 0.$$

Theorem 13. *Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$ $(n \geq 1)$, $\xi \in \partial D$, and let Y be a relatively compact complex space of an Hermitian manifold N with the Hermitian metric ds_N. Let $\{a^m\}$ be a sequence of points in D that tends to*

[6] The article [11] was appeared on March 1986

[7] Received by the Editor on June 16, 1986

a boundary point $\xi \in \partial D$ *at which the unit outward normal* ν_ξ *exists and let* $\lim_{m\to\infty} k_D(a^m, a^{m+1}) = 0$. *If* $f \in H(D,Y)$ *is a normal mapping such that*

$$\lim_{m\to\infty} s_N(f(a^m), l) = 0$$

for some $l \in \overline{Y}$, *then*

$$\lim_{\mathcal{A}_\alpha^\epsilon(\xi)\ni z\to\xi} s_N(f(z), l) = 0 \text{ for all } \alpha > 0.$$

From this we immediately obtain the generalization of the Lindelöf-Lehto-Virtanen's theorem proved by the author [9] in 1982.

Theorem 14. *Let* D *be a strongly pseudoconvex domain in* $\mathbb{C}^n$ *with* C^2*-boundary and let* Y *be a relatively compact complex space of an Hermitian manifold* N *with the Hermitian metric* ds_N. *Let* $f \in H(D,N)$ *be normal in* D *and* $\xi \in \partial D$. *Let* $l \in \overline{Y}$ *and suppose that* f *has the radial limit* l *at* ξ. *Then* f *has the hypoadmissible limit* l *at* ξ.

Results related to Theorem 14 can be found in [1],[6],[21],[22].

Example (Rudin,[31, 8.4.7]) Fix $c > 1/2$. The holomorphic function $f(z_1, z_2) = (1 - z_1)^{-c} z_2 \in H^p(B)$ for all $p < 4/(2c-1)$. The function f has the radial limit at the point $\mathbf{1} = (1,0) \in \partial B$, but does not have a hypoadmissible limit at $\mathbf{1}$. It follows, generally speaking, that the class of normal functions differs from the Hardy H^p-classes even in the case of only holomorphic normal functions.

5 Polynomiality criterion for entire functions

A function $f : \mathbb{C}^n \to \mathbb{C}$, $n \geq 1$, holomorphic on the whole n-dimensional space $\mathbb{C}^n$ is called an *entire function*.

If an entire function f has the homogenious polynomial expansion

$$f(z) = \sum_{j=0}^{\infty} P_j(z),$$

where P_j are homogenious polynomials in $\mathbb{C}^n$ of degree j, then the *radial derivative* Rf is defined as ([31]):

$$\mathcal{R}f(z) = \sum_{j=1}^{\infty} jP_j(z).$$

We prove the "radial" polynomiality criterion for entire functions of several complex variables.

Theorem 15. *An entire function* f, $\mathbb{C}^n$, $n \geq 1$, *is a polynomial if and only if for any complex line* $l \subseteq \mathbb{C}^n$ *passing through the origin we have*

$$\overline{\lim}_{|\lambda|\to\infty} \frac{|\mathcal{R}f(l(\lambda))|}{1 + |f(l(\lambda))|^2} < \infty.$$

References

[1] Aladro G., *Application of the Kobayashi metric to normal functions of several complex variables.* – Util. Math., 1987, **37**, p. 13–24.

[2] Bagemihl F., Seidel W., *Sequential and continuous limits of meromorphic functions.* – Ann. Acad. Sci. Fenn., Ser. AI, 1960, **280**.

[3] Bhenke H., Thullen P., *Theorie der Functionen mehrerer komplexer. Veränderlichen. 2nd ed.,* – Springer-Verlag, Berlin-Heidelberg, 1970.

[4] Bremermann H.J., *Holomorphic continuation of the kernels and the Bergman metric in several complex variables.* – Lectures on Functions of a Complex Variables. Univ. of Michigan Press. Ann Arbor. Mich., 1955, p. 349–383.

[5] Burns D., Shnider S., Wells R., *On deformations of strongly pseudoconvex domains.* – Invent. Math., 1978, **46**, p. 237–253.

[6] Cima J.A., Krantz S.G., *The Lindelöf principle and normal functions of several complex variables.* – Duke Math. J., 1983, **50**, p. 303–328.

[7] Čirca E.M., *The theorems of Lindelöf and Fatou in* $\mathbb{C}^n$. – Mat. sb., 1973, no. 92(134), p. 622 (English transl. in Math U.S.S.R. sb., 1973, **21**, p. 619).

[8] Dovbush P.V., *Normal functions of many complex variables.* – Vestnic Moskov. Univ. Ser. 1, Mat. Meh., 1981, no. 1, p. 44 (English transl. in Moscow Univ. Bull., 1981, **36**).

[9] Dovbush P.V., *The Lindelöf theorem in* $\mathbb{C}^n$. – Vestnic Moskov. Univ. Ser. 1, Mat. Meh., 1981, no. 6, p. 33 (English transl. in Moscow Univ. Bull., 1981, **36**).

[10] Dovbush P.V., *Boundary behavior of normal holomorphic function of several complex variables.* – Dokl. Akad. Nauk SSSR, 1982, **263** p. 14 (English transl. in Soviet Math. Dokl., 1982, **25**).

[11] Dovbush P.V., *About boundary behavior of holomorphic function of several complex variables.* – Mat. Zametki, 1986, no. 39(3) (English transl. in Math. Notes, 1986, **39**).

[12] Dovbush P.V., *About existence of admissible limits of holomorphic function of several complex variables.* – Sibirsk. Mat. Zu., 1987, no. 28(3) (English transl. in Siberian Math. J., 1987, **28**).

[13] Dovbush P.V., *On the Lindelöf theorem in* $\mathbb{C}^n$. – Ukraïn. Mat. Zh., 1988, no. 40(6) (English transl. in Ucrainian Math. J., 1988, **40**).

[14] Dovbush P.V., *About admissible limits of holomorphic function of several complex variables.* – Mat. Zametki, 1990, no. 47(5) (English transl. in Math. Notes, 1990, **47**).

[15] Dovbush P.V., *Polynomiality criterion for entire functions of several complex variables.* – Mat. Zametki, 1999, **47** (English transl. in Math. Notes, 1999, **66**).

[16] Dovbush P.V., *The negligible set for normal functions of several complex variables.* – Rev. Roumaine de Math. Pures et Appl., 2001, **46**, no. 1, p. 29–35.

[17] Dovbush P.V., *Normal functions in convex domains in* $\mathbb{C}^n$. – Mathematica Montisnigri, 2001, **XIV**, p. 79–88 (In Russian).

[18] Dovbush P.V., Gavrilov V.I., *Boundary singularities generated by cluster sets of functions of several complex variables.* – Dokl. Akad. Nauk SSSR, 1982, **265**, p. 1047 (English transl. in Soviet Math. Dokl., 1982, **26**).

[19] Dovbush P.V., Gavrilov V.I., *Normal mappings.* – Mathematica Montisnigri, 2001, **XIV**, p. 5–61 (In Russian).

[20] Funahashi K., *Normal holomorphic mappings and classical theorems of function theory.* – Nagoya Math J., 1984, **94**, p. 89–104.

[21] Hahn K.T., *Asymptotic behavior of normal mappings of several complex variables.* – Canad. J. Math., 1984, **36**, p. 718.

[22] Hahn K.T., *Non-tangential limit theorems for normal mappings.* – Pacific J. Math., 1988, **135**, no. 1, p. 57–64.

[23] Hakim M., Sibony N., *Functions holomorphes bornées et limites tangentielles.* – Duke Math. J., 1983, **50**, no. 1, p. 123–141.

[24] Järvi P., *An extension theorem for normal functions.* – Proc. Amer. Math. Soc., 1988, **103**, p. 1171.

[25] Krantz S.G., *Invariant metrics and the boundary behavior of holomorphic functions on domains in* $\mathbb{C}^n$. – J. Geom. Anal., 1991, **1**, no. 2, p. 71–97.

[26] Lehto O., Virtanen K.I., *Boundary behavior and normal meromorphic functions.* – Acta Math., 1957, **97**, p. 47.

[27] Noshiro K., *Contributions to the theory of meromorphic functions in the unit circle.* – J. Fac. Sci. Hokaido Imp. Univ., Ser. I, 1938, **7**, p. 149.

[28] Ramey W.C., *On the behavior of holomorphic functions near maximum modulus sets.* – Math. Ann., 1986, **276**, p. 137–144.

[29] Riihentaus J., *A nulset for normal functions in several variables.* – Proc. Amer. Math. Soc., 1990, **110**, p. 923.

[30] Rosay I.-P., *Sur une characterization de la boule parmi les domaines de* $\mathbb{C}^n$ *par son groupe d'automorphismes.* – Ann. Inst. Fourier (Grenoble), 1979, **29**, no. 1, p. 91–97.

[31] Rudin W., *Function Theory in the Unit Ball in* $\mathbb{C}^n$. – Springer-Werlag, Berlin, 1983.

[32] Stein E.M., *Boundary behavior of holomorphic function of several complex variables.* – Mathematical Notes, Princeton, NJ: Princeton University Press, 1972.

[33] Timoney R.M., *Bloch function in several complex variables, 1.* – Bull. London Math. Soc., 1980, **12**, no. 37, p. 241–267.

[34] Wong B., *Characterization of the unit ball in* $\mathbb{C}^n$ *by its automorphism group.* – Invent. Math., 1977, **41**, p. 253–257.

[35] Wu H., *Normal families of holomorphic mappings.* – Acta Math., 1967, **119**, no. 3-4, p. 193–233.

[36] Yosida K., *On a class of meromorphic functions.* – Proc. Phys.-Math. Soc. Japan, 3 ser., 1934, **16**, p. 235.

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