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Invariant conditions for the dimensions of the $GL(2,R)$-orbits for one differential cubic system

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Abstract. A two-dimensional system of two autonomous polynomial equations with homogeneities of the zero and third orders is considered concerning to the group of center-affine transformations $GL(2,R)$. The problem of the classification of $GL(2,R)$-orbit's dimensions is solved completely for the given system with the help of Lie algebra of operators corresponding to the $GL(2,R)$ group, and algebra of invariants and comitants for the indicated system is built. The theorem on invariant division of all coefficient's set of the considered system to nonintersecting $GL(2,R)$-invariant sets is obtained.

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Consider the differential system

$$\frac{dx}{d\tau} = a + px^3 + 3qx^2y + 3rxy^2 + sy^3,\ \frac{dy}{d\tau} = b + tx^3 + 3ux^2y + 3vxy^2 + wy^3, \quad (1)$$

where the coefficients and variables take values from the field of real numbers R.

Let $A = (a, b, p, q, r, s, t, u, v, w) \in E(A)$, where $E(A)$ is the Euclidean space of the coefficients of right-hand sides of the system (1).

Will denote by $A(T)$ the point from $E(A)$ that belongs to the system, obtained from the system (1) with coefficients A by transformation $T \in GL(2,R)$.

Definition 1. *The set $O(A) = \{A(T); T \in GL(2,R)\}$ is called $GL(2,R)$–orbit of the point A for the system (1).*

Definition 2. *Call the set $M \subseteq E(A)$ $GL(2,R)$-invariant if for any point $A \in M$ its orbit $O(A) \subseteq M$.*

It is known (see, for instance, [1]), that

$$dim_R O(A) = rank M_1,$$

where M_1 is the matrix is constructed on the coordinate vectors of the Lie algebra operators obtained as a result of the representation of the $GL(2,R)$ group in the space $E(A)$ of the system (1).

With the help of [1] it is possible to find that the matrix M_1 has the form

$$M = \begin{pmatrix} -a & 0 & 2p & q & 0 & -s & 3t & 2u & v & 0 \\ -b & 0 & -t & p-u & 2q-v & 3r-w & 0 & t & 2u & 3v \\ 0 & -a & 3q & 2r & s & 0 & 3u-p & 2v-q & w-r & -s \\ 0 & -b & 0 & q & 2r & 3s & -t & 0 & v & 2w \end{pmatrix}. \tag{2}$$

Consider the invariants and comitants of the system (1) with respect to the group $GL(2,R)$, found in [2-3], which will be used further. With this purpose we rewrite the system (1) in the tensor form according to [4]

$$\frac{dx^j}{dt} = a^j + a^j_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma, \quad (j,\alpha,\beta,\gamma = 1,2), \tag{3}$$

where coefficient tensor $a^j_{\alpha\beta\gamma}$ is symmetrical in lower indexes, in which the complete convolution takes place. Note that among the coefficients and variables of the systems (1) and (3) there are equalities

$$x^1 = x,\ a^1 = a,\ a^1_{111} = p,\ a^1_{112} = q,\ a^1_{122} = r,\ a^1_{222} = s$$

$$x^2 = y,\ a^2 = b,\ a^2_{111} = t,\ a^2_{112} = u,\ a^2_{122} = v,\ a^2_{222} = w. \tag{4}$$

Then needed by us comitants and invariants of the system (3), and, consequently, of the system (1), take the form

$$P_1 = a^\alpha_{\alpha\beta\gamma}x^\beta x^\gamma,\ P_2 = a^p_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma x^q \epsilon_{pq},\ P_3 = a^\alpha_{p\alpha\beta}a^\beta_{q\gamma\delta}x^\gamma x^\delta \epsilon^{pq},$$

$$P_4 = a^\alpha_{\alpha\beta\gamma}a^\beta_{\delta\mu\theta}x^\gamma x^\delta x^\mu x^\theta,\ P_5 = a^\alpha_{\beta\gamma\delta}a^\beta_{\alpha\mu\theta}x^\gamma x^\delta x^\mu x^\theta,$$

$$p_2 = a^p_{\alpha\beta\gamma}a^\alpha x^\beta x^\gamma x^q \epsilon_{pq},\ p_9 = a^\alpha_{\beta\gamma\delta}a^\beta_{\alpha\mu\nu}a^\gamma x^\delta x^\mu x^\nu,\ p_{27} = a^p x^q \epsilon_{pq}$$

$$J_1 = a^\alpha_{\alpha pr}a^\beta_{\beta qs}\epsilon^{pq}\epsilon^{rs},\ J_2 = a^\alpha_{\beta pr}a^\beta_{\alpha qs}\epsilon^{pq}\epsilon^{rs},\ J_4 = a^\alpha_{pru}a^\beta_{\gamma qs}a^\gamma_{\alpha\beta v}\epsilon^{pq}\epsilon^{rs}\epsilon^{uv}, \tag{5}$$

where $\varepsilon^{pq}(\varepsilon^{11} = \varepsilon^{22} = 0,\quad \varepsilon^{12} = -\varepsilon^{21} = 1$ and $\varepsilon_{pq}(\varepsilon_{11} = \varepsilon_{22} = 0,\ \varepsilon_{12} = -\varepsilon_{21} = 1)$ are unit bivectors.

Considering (4) and (5) it is easy to establish the following

Remark 1. *The condition $p_{27} \equiv 0$ for the system (1) is equivalent to the equalities*

$$a = b = 0. \tag{6}$$

Taking into account Remark 1, Theorem 1.44 and Lemma 1.44 from [1] it is easy to obtain

Lemma 1. *If $p_{27} \equiv 0$ the rank of matrix (2) is equal to*

4 for $P_1P_2(3P_1P_3 - 2J_1P_2) \not\equiv 0$, *or*
50 $P_1 \equiv 0,\ P_2(J_2P_5 - J_4P_2) \not\equiv 0$;
3 for $P_1P_2 \not\equiv 0,\ 3P_1P_3 - 2J_1P_2 \equiv 0$, *or*
$P_2P_5 \not\equiv 0,\ P_1 \equiv J_2P_5 - J_4P_2 \equiv 0$, *or*

$P_2 \equiv 0,\ J_1 \neq 0$;
2 for $P_2 \not\equiv 0,\ P_1 \equiv J_2P_5 - J_4P_2 \equiv P_5 \equiv 0$, or
$P_2 \equiv 0,\ J_1 = 0,\ P_1 \not\equiv 0$;
0 for $P_1 \equiv P_2 \equiv 0$,
where P_1, P_2, P_3, P_5, J_1, J_2, J_4 are taken from (5).

Let us prove

Lemma 2. *If $P_2 \equiv 0$ the rank of matrix (2) is equal to*
4 for $J_1p_{27} \not\equiv 0$; (7)
3 for $P_1 \not\equiv 0,\ p_{27}J_1 \equiv 0,\ p_{27} + J_1 \not\equiv 0$; (8)
2 for $P_1 \not\equiv 0, p_{27} \equiv 0, J_1 = 0, or P_1 \equiv 0,\ p_{27} \not\equiv 0$; (9)
0 for $P_1 \equiv p_{27} \equiv 0$, (10)
where P_1, P_2, p_{27}, J_1 are taken from (5).

Proof. Consider two cases: 1) If $p_{27} \equiv 0$, owing to the fact that $J_1 \neq 0$ implies that $P_1 \not\equiv 0$ (see [1]), we obtain that the corresponding cases of Lemma 2 coincide with the corresponding cases of Lemma 1, and, hence, its truth is evident.

2) Let $p_{27} \not\equiv 0$, i.e., according to (4)-(5), we have

$$a^2 + b^2 \neq 0. \tag{11}$$

Since $P_2 \equiv 0$ from (4)-(5) we obtain for the system (1)

$$t = 0, \quad p = 3u, \quad q = v, \quad w = 3r, \quad s = 0. \tag{12}$$

Because of (12), removing the zero columns matrix (2) takes the form

$$M_1^{(1)} = \begin{pmatrix} -a & 0 & 0 & 2u & v \\ -b & 0 & v & 0 & 2u \\ 0 & -a & 0 & v & 2r \\ 0 & -b & 2r & 0 & v \end{pmatrix}. \tag{13}$$

Consider the following subcases:

a) Denote by the Δ_{ijkl} $(1 \leq i, j, k, l \leq 5)$ every possible minors of the fourth order of the matrix $M_1^{(1)}$ constructed on its columns with the numbers i, j, k, l. There is no difficulty to see that the following minors will be different from zero

$$8\Delta_{1234} = -abJ_1, \quad 8\Delta_{1235} = a^2J_1, \quad 8\Delta_{1245} = -b^2J_1,$$

$$8\Delta_{1345} = (av + 2br)J_1, \quad 8\Delta_{2345} = -(bv + 2au)J_1, \tag{14}$$

where J_1 is invariant from (5), having in this case for the system (1) the form

$$J_1 = -8(v^2 - 4ur). \tag{15}$$

Taking into account (11) and (14) we note that the rank of matrix $M_1^{(1)}$ is equal to 4 if and only if the condition (7) takes place.

b) Due to (14) and (15) we can say that if with $p_{27} \not\equiv 0$ takes place the condition

$$J_1 = 0 \ \ (v^2 = 4ur), \tag{16}$$

then all minors of the fourth order of the matrix (11) are equal to zero and, hence, $rankM_1^{(1)} < 4$.

There is no difficulty to check that from 40 different minors of the third order of the matrix (13) the following will be different from zero if (16) takes place:

$$\Delta_{123}^{123} = -a^2v,\ \Delta_{124}^{123} = 2abu,\ \Delta_{125}^{123} = abv - 2a^2u,\ \Delta_{245}^{123} = -4au^2,$$

$$\Delta_{145}^{123} = \Delta_{234}^{123} = \Delta_{245}^{134} = 2auv,\ \Delta_{124}^{124} = 2b^2u,\ \Delta_{125}^{124} = b^2v - 2abu,$$
$$\Delta_{134}^{124} = -4bur,\ \Delta_{135}^{124} = -\Delta_{134}^{234} = \Delta_{235}^{234} = -2bvr,\ \Delta_{245}^{124} = -4bu^2,$$
$$\Delta_{123}^{134} = 2a^2r,\ \Delta_{125}^{134} = a^2v - 2abr,\ \Delta_{135}^{134} = 4ar^2,\ \Delta_{234}^{134} = -4aru,$$
$$\Delta_{123}^{234} = 2abr,\ \Delta_{124}^{234} = -b^2v,\ \Delta_{125}^{234} = abv - 2b^2r,\ \Delta_{135}^{234} = 4br^2,$$
$$\Delta_{134}^{123} = -\Delta_{235}^{123} = \Delta_{145}^{134} = -av^2,\ \Delta_{135}^{123} = -\Delta_{134}^{134} = \Delta_{235}^{134} = -2arv,$$
$$\Delta_{123}^{124} = \Delta_{124}^{134} = -abv,\ \Delta_{235}^{124} = -\Delta_{145}^{234} = -\Delta_{234}^{234} = bv^2,$$
$$\Delta_{145}^{124} = \Delta_{234}^{124} = \Delta_{245}^{234} = 2buv, \tag{17}$$

where Δ_{lmn}^{ijk} $(1 \le i,j,k \le 4; 1 \le l,m,n \le 5)$ are indicated minors of the matrix $M_1^{(1)}$ constructed on lines i,j,k and columns l,m,n.

As with the help of (4)-(5) and (12) for P_1 we obtain

$$P_1 = 4ux^2 + 4vxy + 4ry^2, \tag{18}$$

then with (11) we have that at least one of minors (17) is different from zero if and only if $P_1p_{27} \not\equiv 0$ in this subcase. Therefore (8) is true. Let us note that $J_1 = 0$ does not contradict to $P_1 \not\equiv 0$.

c) It follows from (17) and (18) that if with $p_{27} \not\equiv 0$ the equality

$$P_1 \equiv 0 \ \ (u = v = r = 0) \tag{19}$$

takes place then all minors of the third order of matrix (13) are equal to zero and hence $rankM_1^{(1)} < 3$.

Let form all possible unzero minors of the second order of matrix (13), which will denote by $\Delta_{kl}^{ij}(1 \le i,j \le 4; 1 \le k,l \le 2)$. It is not difficult to see that they are the following:

$$\Delta_{12}^{13} = a^2,\ \Delta_{12}^{14} = \Delta_{12}^{23} = ab,\ \Delta_{12}^{24} = b^2. \tag{20}$$

With (11) at least one of minors (20) is different from zero and hence the rank of matrix (13) is equal to 2. And this provides the fulfillment of the second condition from (9).

The case (10) is evident. Lemma 2 is proved.

Lemma 3. *If $P_1P_2 \not\equiv 0$ the rank of matrix (2) is equal to 4 if and only if*

$$2P_2(3P_1P_3 - 2J_1P_2) + W_1 \not\equiv 0, \tag{21}$$

where

$$W_1 = 3P_2(P_3p_{27} + 2p_9) + 2P_5(P_1p_{27} - 4p_2), \tag{22}$$

and $P_1, P_2, P_3, P_5, p_2, p_9, p_{27}, J_1$ are taken from (5).

Proof. Necessity. Similarly to the proof of Lemma 2.44 from [1] we consider 3 cases.

1) The discriminant $D(P_1) > 0$. Then, taking into consideration the expression for P_1 from (4)-(5) it is easy to check that by the center-affine transformation [1] we obtain

$$P_1 = 2(q+v)xy,\ p = -u,\ r = -w, \tag{23}$$

where

$$q + v \neq 0. \tag{24}$$

Let assume that the condition (21) is not necessary, i.e. that $2P_2(3P_1P_3 - 2J_1P_2) + W_1 \equiv 0$. Since the expressions $2P_2(3P_1P_3 - 2J_1P_2)$ and W_1 have 8th and 7th degrees, respectively, concerning variables x, y, then the last identity is equivalent to the system

$$2P_2(3P_1P_3 - 2J_1P_2) \equiv 0, \tag{25}$$

$$W_1 \equiv 0. \tag{26}$$

Taking into consideration the expression for P_1 from (4)-(5), conditions (23), (25), we obtain the following values for the coefficients of the system (1)

$$p = r = s = t = u = w = 0. \tag{27}$$

With these coefficients the comitant P_2 and the identity (26) take the form

$$P_2 = 3(q-v)x^2y^2,$$

$$W_1 = -6b(q-v)(-q^2 + qv + 5v^2)x^4y^3 + 6a(q-v)(-5q^2 - qv + v^2)x^3y^4 \equiv 0.$$

From the last identity with $P_2 \not\equiv 0$ we obtain for the coefficients of the system (1) as real values that $a = b = 0$. In this case removing the zero columns matrix (2) takes the form

$$M_1^{(2)} = \begin{pmatrix} 0 & q & 0 & 0 & v & 0 \\ 0 & 0 & 2q-v & 0 & 0 & 3v \\ 3q & 0 & 0 & 2v-q & 0 & 0 \\ 0 & q & 0 & 0 & v & 0 \end{pmatrix}.$$

There is no difficulty to see that all the minors of the fourth order of the matrix $M_1^{(2)}$ are equal to zero, i.e. $rank M_1^{(2)} < 4$.

Obtained contradictions prove the necessity of the condition (21).

2) The discriminant $D(P_1) = 0$. Then, taking into consideration the expression for P_1 from (4)-(5) it is easy to check that by the center-affine transformation [1] we obtain

$$P_1 = (p+u)x^2, \; q = -v, \; r = -w, \tag{28}$$

where

$$p + u \neq 0. \tag{29}$$

Let assume that the condition (21) is not necessary, i.e. let consider (25)-(26). Taking into account (25), (28), (29), we obtain the following values for the coefficients of the system (1):

$$q = r = s = v = w = 0. \tag{30}$$

With these coefficients the comitant P_2 and the identity (26) take the form

$$P_2 = -tx^4 + (p - 3u)x^3y,$$

$$W_1 = 2(p^2 + u^2)(at - bp + 3bu)x^7 \equiv 0.$$

From the last identity with (29) and $P_2 \not\equiv 0$ we obtain the following real values for the coefficients of the system (1):

a) $a = \frac{b(p-3u)}{t}$, $(t \neq 0)$. In this case removing zero columns the matrix (2) takes the form

$$M_1^{(3)} = \begin{pmatrix} \frac{b(3u-p)}{t} & 0 & 2p & 0 & 3t & 2u & 0 \\ -b & 0 & -t & p-u & 0 & t & 2u \\ 0 & \frac{b(3u-p)}{t} & 0 & 0 & 3u-p & 0 & 0 \\ 0 & -b & 0 & 0 & -t & 0 & 0 \end{pmatrix}.$$

There is no difficulty to see that all the minors of the fourth order of the matrix $M_1^{(3)}$ are equal to zero, i.e. $rankM_1^{(3)} < 4$, that proves the necessity of the conditions (21).

b) $b = t = 0$. In this case removing zero columns the matrix (2) takes the form

$$M_1^{(4)} = \begin{pmatrix} -a & 0 & 2p & 0 & 0 & 2u & 0 \\ 0 & 0 & 0 & p-u & 0 & 0 & 2u \\ 0 & -a & 0 & 0 & 3p-u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

There is no difficulty to see that all the minors of the fourth order of the matrix $M_1^{(4)}$ are equal to zero, i.e. $rankM_1^{(3)} < 4$, that proves the necessity of the conditions (21).

3) The discriminant $D(P_1) < 0$. Then, taking into consideration the expression for P_1 from (4)-(5) it is easy to check that by the center-affine transformation [1] we obtain

$$P_1 = A(x^2 + y^2) \not\equiv 0, \; A = p + u = r + w. \tag{31}$$

Let assume that the condition (21) is not necessary, i.e. let consider (25)-(26). Taking into account (25) and (31), we obtain the following values for the coefficients of the system (1):

$$p = w = 3A/4,\ q = -v,\ r = u = A/4,\ s = -t = -3v.$$

With these coefficients the comitant P_2 and the identity (26) take the form

$$P_2 = -3v(x^2 + y^2)^2;$$

$$W_1 = -\frac{3v}{4}[(48av^2 + 48bvA - 5aA^2)x + (48bv^2 - 48avA - 5bA^2)y](x^2 + y^2)^3 \equiv 0.$$

Taking into consideration the last identity with $P_2 \not\equiv 0$ we obtain the following values for the coefficients of the system (1):

$$a = b = 0, (v \neq 0).$$

In this case removing zero columns the matrix (2) takes the form

$$M_1^{(5)} = \begin{pmatrix} \frac{3A}{2} & -v & 0 & 3v & 9v & \frac{A}{2} & v & 0 \\ -3v & \frac{A}{2} & -3v & 0 & 0 & 3v & \frac{A}{2} & 3v \\ -3v & \frac{A}{2} & -3v & 0 & 0 & 3v & \frac{A}{2} & 3v \\ 0 & -v & \frac{A}{2} & -9v & -3v & 0 & v & \frac{3A}{2} \end{pmatrix}.$$

Note that the second and third lines of the matrix $M_1^{(5)}$ coincide, hence $rank M_1^{(5)} < 4$, that proves the necessity of the conditions (21).

The necessity of the conditions (21) is proved completely.

Sufficiency of the conditions (21) with $P_1P_2 \not\equiv 0$ follows from the expression $2P_2(3P_1P_3 - 2J_1P_2)$ written by the minors of the matrix (2), see [1], p.164; and the expression

$$W_1 = (-2\Delta_{1378} + \Delta_{2347} - \Delta_{2379} - \Delta_{2478})x^7 + (6\Delta_{1347} - 3\Delta_{1379} - 14\Delta_{1478} - 18\Delta_{1789} -$$
$$-7\Delta_{2348} + 10\Delta_{2357} + \Delta_{23710} - 12\Delta_{2389} - 11\Delta_{2479} + 4\Delta_{2578} - 5\Delta_{27810})x^6y +$$
$$+(-5\Delta_{1348} - 29\Delta_{1357} - 4\Delta_{13710} + 23\Delta_{1479} - 13\Delta_{1578} + 18\Delta_{17810} + 16\Delta_{2349} + 10\Delta_{2358} -$$
$$-21\Delta_{2367} - 15\Delta_{2457} + \Delta_{24710} + 33\Delta_{2489} - 7\Delta_{2579} - 22\Delta_{2678} + 10\Delta_{27910})x^5y^2 +$$
$$+(-37\Delta_{1349} - 10\Delta_{1358} + 8\Delta_{1367} + 24\Delta_{1457} - 96\Delta_{1489} + 13\Delta_{1579} + 9\Delta_{1678} - 14\Delta_{17910} -$$
$$-15\Delta_{2345} + \Delta_{23410} - 37\Delta_{2359} + 31\Delta_{2368} - 10\Delta_{23910} - 48\Delta_{2458} + 27\Delta_{2467} - 38\Delta_{24810} -$$
$$-9\Delta_{25710} - 33\Delta_{2589} + 25\Delta_{2679} - 6\Delta_{28910})x^4y^3 + (-6\Delta_{1345} - 10\Delta_{13410} - 38\Delta_{1359} -$$
$$-9\Delta_{1368} + \Delta_{13910} - 33\Delta_{1458} + 25\Delta_{1467} - 37\Delta_{14810} + 31\Delta_{15710} - 48\Delta_{1589} + 27\Delta_{1679} -$$
$$-15\Delta_{18910} - 14\Delta_{2346} - 96\Delta_{2459} + 13\Delta_{2468} - 37\Delta_{24910} + 9\Delta_{2567} - 10\Delta_{25810} + 8\Delta_{26710} +$$
$$+24\Delta_{2689})x^3y^4 + (10\Delta_{1346} + \Delta_{1369} + 33\Delta_{1459} - 7\Delta_{1468} + 16\Delta_{14910} - 22\Delta_{1567} +$$

$$+10\Delta_{15810}-21\Delta_{16710}-15\Delta_{1689}+18\Delta_{2356}-4\Delta_{23610}+23\Delta_{2469}-13\Delta_{2568}-5\Delta_{25910}-$$
$$-29\Delta_{26810})x^2y^5+(-5\Delta_{1356}+\Delta_{13610}-12\Delta_{14510}-11\Delta_{1469}+4\Delta_{1568}-7\Delta_{15910}+$$
$$+10\Delta_{16810}-18\Delta_{2456}-3\Delta_{24610}-14\Delta_{2569}+6\Delta_{26910})xy^6+(-\Delta_{14610}-\Delta_{1569}+\Delta_{16910}-$$
$$-2\Delta_{25610})y^7, \qquad (32)$$

where Δ_{ijkl} $(1 \le i,j,k,l \le 10)$ is the minor of the fourth order of the matrix (2) constructed on its columns with the numbers i,j,k,l. Lemma 3 is proved.

Lemma 4. *If $P_1 \equiv 0,\ P_2 \not\equiv 0$ the rank of the matrix (2) is equal to 4 if and only if*

$$J_2P_5 - J_4P_2 + W_1 + W_2 \not\equiv 0, \qquad (33)$$

where W_1 is taken from (22), and

$$W_2 = p_{27}^2(P_1^2 + 6P_4 - 9P_5) + 2p_2^2, \qquad (34)$$

where $P_1, P_2, P_4, P_5, p_2, p_{27}, J_2, J_4$ are taken from (5).

Proof. Necessity. From $P_1 \equiv 0$ we obtain the following values for the coefficients of the system (1):

$$p=-u,\ q=-v,\ r=-w. \qquad (35)$$

With coefficients (35) comitants P_2, W_1 and W_2 take the form

$$P_2 = -tx^4 - 4ux^3y - 6vx^2y^2 - 4wxy^3 + sy^4, \qquad (36)$$

$$W_1 = (-4at^2v + 4atu^2 + 6bt^2w - 22btuv + 16bu^3)x^7 + (-14at^2w + 14atuv - 6bst^2+$$
$$+4btuw - 66btv^2 + 56bu^2v)x^6y + (10ast^2 - 44atuw + 54atv^2 - 26bstu - 126btvw+$$
$$+64bu^2w + 36buv^2)x^5y^2 + (50astu + 70atvw - 80au^2w + 60auv^2 + 16bstv - 56bsu^2-$$
$$-84btw^2 + 8buvw + 36bv^3)x^4y^3 + (16astv + 84asu^2 + 56atw^2 + 8auvw + 36av^3+$$
$$+50bstw - 70bsuv - 80buw^2 + 60bv^2w)x^3y^4 + (-26astw + 126asuv + 64auw^2+$$
$$+36av^2w - 10bs^2t + 44bsuw - 54bsv^2)x^2y^5 + (6as^2t - 4asuw + 66asv^2 + 56avw^2-$$
$$-14bs^2u - 14bsvw)xy^6 + (6as^2u + 22asvw + 16aw^3 - 4bs^2v - 4bsw^2)y^7, \qquad (37)$$

and

$$W_2 = (2a^2t^2 + 4abtu + 18b^2tv - 16b^2u^2)x^6 + (12a^2tu - 24abtv + 48abu^2 + 36b^2tw-$$
$$-24b^2uv)x^5y + (30a^2tv - 60abtw + 120abuv - 18b^2st + 48b^2uw - 36b^2v^2)x^4y^2+$$
$$+(40a^2tw + 32abst - 32abuw + 144abv^2 - 40b^2su)x^3y^3 + (-18a^2st + 48a^2uw-$$
$$-36a^2v^2 + 60absu + 120abvw - 30b^2sv)x^2y^4 + (-36a^2su - 24a^2vw + 24absv+$$
$$+48abw^2 - 12b^2sw)xy^5 + (-18a^2sv - 16a^2w^2 - 4absw + 2b^2s^2)y^6. \qquad (38)$$

Let assume that the condition (33) is not necessary i.e. that if $J_2P_5 - J_4P_2 + W_1 + W_2 \equiv 0$, than there are some minors of the fourth order of the matrix (2) which are different from zero. As the expressions $J_2P_5 - J_4P_2$, W_1 and W_2 have 4th, 7th and 6th degrees, respectively, concerning variables x, y, then the last identity is equivalent to the system

$$J_2P_5 - J_4P_2 \equiv W_1 \equiv W_2 \equiv 0. \tag{39}$$

With (35) from the first identity from (39) we obtain the system of the polynomial equations of the fourth degree concerning coefficients of the system (1). Solving indicated system (see [1], ? 171), we obtain the following four real solutions:

$$1)\ p = q = r = s = t = u = v = w = 0. \tag{40}$$

With these coefficients $P_2 \equiv 0$, what contradicts to the condition of Lemma 4.

$$2)\ p = q = t = u = v = 0,\ r = -w. \tag{41}$$

With these equalities unzero minors of the fourth order of the matrix (2) will be the following

$$\Delta_{16910} = \Delta_{1569} = \Delta_{14610} = \Delta_{1456} = -2\Delta_{1269} = -2\Delta_{1246} = 4w^2(4aw - bs),$$

$$\Delta_{12610} = \Delta_{1256} = 8a^2w^2 + 2absw - b^2s^2. \tag{42}$$

And expressions W_1 and W_2 take the form

$$W_1 = 4w^2(4aw - bs)y^7,$$

$$W_2 = 12bw(4aw - bs)xy^5 + 2(-8a^2w^2 - 2absw + b^2s^2)y^6. \tag{43}$$

Taking into consideration the last equalities from (39) and the obtained with the help (42)-(43) contradiction we find the necessity of the conditions (33) in this case, too.

$$3)\ p = t = u = 0,\ q = -v,\ r = -w,\ 3sv = -2w^2. \tag{44}$$

Let substitute these coefficients into W_1 and W_2:

$$W_1 = 36bv^3x^4y^3 + (60bv^2w + 36av^3)x^3y^4 + (36av^2w + 36bvw^2)x^2y^5 + (12avw^2+$$

$$+\frac{28}{3}bw^3)xy^6 + (\frac{4}{3}aw^3 - \frac{4}{3}bsw^2)y^7;$$

$$W_2 = 36b^2v^2x^4y^2 + 144abv^2x^3y^3 + (-36a^2v^2 + 120abvw + 20b^2w^2)x^2y^4+$$

$$+(-24a^2vw + 32abw^2 - 12b^2sw)xy^5 + (-4a^2w^2 - 4absw + 2b^2s^2)y^6.$$

From $W_1 \equiv W_2 \equiv 0$ with the last equalities we obtain:

a) $a = b = 0, (v \neq 0)$. Matrix (2) takes the form

$$M_1^{(6)} = \begin{pmatrix} 0 & 0 & 0 & -v & 0 & -s & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & -3v & -4w & 0 & 0 & 0 & 3v \\ 0 & 0 & -3v & -2w & s & 0 & 0 & 3v & 2w & -s \\ 0 & 0 & 0 & -v & -2w & 3s & 0 & 0 & v & 2w \end{pmatrix}.$$

There is no difficulty to see that all the minors of the fourth order of this matrix are equal to zero, i.e. $rank M_1^{(6)} < 4$. Therefore the conditions (33) are necessary in this case.

b) With $v = 0$ from (44) we obtain $p = q = r = t = u = v = w = 0$. With these equalities the following minors of the fourth order of the matrix (2) will be different from zero

$$\Delta_{1256} = \Delta_{12610} = -b^2 s^2, \tag{45}$$

and the expressions W_1 and W_2 take the form

$$W_1 \equiv 0, \ W_2 = 2b^2 s^2 y^6. \tag{46}$$

If we demand that the equality $W_2 \equiv 0$ from (46) takes place, then with the help of (45) we obtain that in this subcase all the minors of the fourth order of the matrix (2) are equal to zero. This contradiction proves the necessity of the condition (33).

$$4) \ t = 0, \ 4uw = 3v^2, \ 2su^2 = -v^3. \tag{47}$$

Let substitute these coefficients into W_1 and W_2:

$$W_1 = 16bu^3x^7 + 56bu^2vx^6y + 112bu^2wx^5y^2 + (-128bsu^2 + 8buvw)x^4y^3 + (12asu^2 + $$
$$+8auvw - 70bsuv)x^3y^4 + (126asuv + 84av^2w - 28bsuw)x^2y^5 + (63asv^2 + 56avw^2 -$$
$$-14bs^2u - 14bsvw)xy^6 + (6as^2u + 22asvw + 16aw^3 - 4bs^2v - 4bsw^2)y^7;$$
$$W_2 = -16b^2u^2x^6 + (48abu^2 - 24b^2uv)x^5y + 120abuvx^4y^2 + (160abuw - 40b^2su)x^3y^3 +$$
$$+(60absu + 120abvw - 30b^2sv)x^2y^4 + (-36a^2su - 24a^2vw + 24absv + 48abw^2 -$$
$$-12b^2sw)xy^5 + (-18a^2sv - 16a^2w^2 - 4absw + 2b^2s^2)y^6.$$

From $W_1 \equiv W_2 \equiv 0$ with the last equalities we obtain $bu = bv = 0$.

If $b \neq 0$, then we come to the case 1) from the proof of the necessity in Lemma 4.

If $b = 0$ then we have:

$$W_1 = 4au(2vw + 3su)x^3y^4 + 42av(2vw + 3su)x^2y^5 + 7av(9sv + 8w^2)xy^6 + 2a(3s^2u +$$
$$+11svw + 8w^3)y^7;$$
$$W_2 = -12a^2(2vw + 3su)xy^5 - 2a^2(8w^2 + 9sv)y^6.$$

From $W_1 \equiv W_2 \equiv 0$ we obtain the following subcases:

a) $a=0$. This subcase is considered in ([1], ?.173).

b) $a\neq 0, s\neq 0, p=q=r=u=v=w=0$. Taking into account the case 3) b) from the proof of the necessity in Lemma 4 we conclude that all the minors of the fourth order are equal to zero here, that proves the necessity of the condition (33).

c) $a\neq 0, u\neq 0, q=r=s=v=w=0$. The matrix (2) in this case takes the form

$$M_1^{(7)}=\begin{pmatrix} -a & 0 & -2u & 0 & 0 & 0 & 0 & 2u & 0 & 0\\ 0 & 0 & 0 & -2u & 0 & 0 & 0 & 0 & 2u & 0\\ 0 & -a & 0 & 0 & 0 & 0 & 4u & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

There is no difficulty to see that all the minors of the fourth order of the matrix $M_1^{(7)}$ are equal to zero, i.a. $rankM_1(7)<4$. Hence the condition (33) is necessary in this subcase.

d) $a\neq 0, v\neq 0, w\neq 0, u=\frac{3v^2}{4w}, s=-\frac{8w^2}{9v}$. Matrix (2) takes the form in this case

$$M_1^{(8)}=\begin{pmatrix} -a & 0 & -\frac{3v^2}{2w} & -v & 0 & \frac{8w^2}{9v} & 0 & \frac{3v^2}{2w} & v & 0\\ 0 & 0 & 0 & -\frac{3v^2}{2w} & -3v & -4w & 0 & 0 & \frac{3v^2}{2w} & 3v\\ 0 & -a & -3v & -2w & -\frac{8w^2}{9v} & 0 & \frac{3v^2}{w} & 3v & 2w & \frac{8w^2}{9v}\\ 0 & 0 & 0 & -v & -2w & -\frac{8w^2}{3v} & 0 & 0 & v & 2w \end{pmatrix}.$$

There is no difficulty to see that all the minors of the fourth order of the matrix $M_1^{(8)}$ are equal to zero, i.a. $rankM_1(8)<4$. Hence the condition (33) is necessary in this subcase.

The necessity of the condition (33) is proved completely.

Sufficiency of the condition (33) with $P_1\equiv 0, P_2\not\equiv 0$ follows from the expression $J_2P_5-J_4P_2$ written by the minors of the matrix (2), see [1], p.169, and the expression

$$W_2=(-\Delta_{1237}-\Delta_{1278})x^6+(-2\Delta_{1238}-4\Delta_{1247}-2\Delta_{1279})x^5y+(-\Delta_{1234}-\Delta_{1239}-$$
$$-9\Delta_{1248}-5\Delta_{1257}-\Delta_{12710}-3\Delta_{1289})x^4y^2+(-2\Delta_{1235}-6\Delta_{1249}-12\Delta_{1258}-2\Delta_{1267}-$$
$$-2\Delta_{12810})x^3y^3+(-\Delta_{1236}-3\Delta_{1245}-\Delta_{12410}-9\Delta_{1259}-5\Delta_{1268}-\Delta_{12910})x^2y^4+$$
$$+(-2\Delta_{1246}-\Delta_{12510}-4\Delta_{1269})xy^5+(-\Delta_{1256}-\Delta_{12610})y^6,$$

where $\Delta_{ijkl}, (1\leq i,j,k,l\leq 10)$- is the minor of the fourth order of the matrix (2), constructed on its columns with the numbers i,j,k,l. Lemma 4 is proved.

Lemma 5. *If $P_1P_2\not\equiv 0$ the rank of the matrix (2) is equal to 3 if and only if*

$$2P_2(3P_1P_3-2J_1P_2)+W_1\equiv 0, \tag{48}$$

where P_1,P_2,P_3,J_1 are taken from (5), and W_1 from (22).

Proof. Necessity of the conditions (48) follows from Lemma 3.

Sufficiency of the conditions (48) can be proved similarly to the first part of the proof of the sufficiency of Lemma 3.44 from [1]. Lemma 5 is proved.

Lemma 6. *If $P_1 \equiv 0, P_2 \not\equiv 0$ the rank of the matrix (2) is equal to 3 if and only if*

$$J_2P_5 - J_4P_2 + W_1 + W_2 \equiv 0, P_5 \not\equiv 0, \tag{49}$$

where P_1, P_2, P_5, J_2, J_4 are taken from (5), W_1 from (22), and W_2 from (34).

Proof. The necessity of the identity (49) follows from Lemma 4. The necessity of the inequality from (49) can be proved similarly to the second part of the proof of the necessity and sufficiency of Lemma 3.44 from [1].

Sufficiency of the conditions (49) can be proved similarly to the second part of the proof of the necessity and sufficiency of Lemma 3.44 from [1]. Lemma 6 is proved.

Lemma 7. *If $P_2 \not\equiv 0$ the rank of the matrix (2) is equal to 2 if and only if*

$$P_1 \equiv P_5 \equiv J_2P_5 - J_4P_2 + W_1 + W_2 \equiv 0, \tag{50}$$

where P_1, P_2, P_5, J_2, J_4 are taken from (5), W_1 from (22), and W_2 from (34).

Proof. Necessity of the condition (50) follows from Lemma 6.

Let prove the sufficiency. If the conditions of Lemma 7 take place, than in every case 1)-10) from the proof of the sufficiency of Lemma 6, where we do not have any contradictions, we obtain $P_2 = sy^4$, and $a = b = p = q = r = t = u = v = w = 0$. With these equalities from $P_2 \not\equiv 0$ follows $s \neq 0$ and the rank of matrix (2) is equal to 2, since the minors of the second order $3\Delta^{13}_{56} = 3\Delta^{13}_{610} = \Delta^{34}_{56} = \Delta^{34}_{610} = 3s^2$ are different from zero. Lemma 7 is proved.

Theorem 1. *$GL(2,R)$-orbit of the system (1) has the dimension*

4 for $P_1P_2 \not\equiv 0,\ 3P_1P_3 - 2J_1P_2 + W_1 \not\equiv 0,$ *or*
$p_{27} \equiv 0,\ P_1P_2(3P_1P_3 - 2J_1P_2) \not\equiv 0,$ *or*
50 $p_{27} \equiv 0,\ P_1 \equiv 0,\ P_2(J_2P_5 - J_4P_2) \not\equiv 0,$ *or*
$P_1 \equiv 0,\ P_2 \not\equiv 0,\ J_2P_5 - J_4P_2 + W_1 + W_2 \not\equiv 0,$ *or*
50 $P_2 \equiv 0,\ J_1p_{27} \not\equiv 0;$

3 for $P_1P_2 \not\equiv 0,\ 3P_1P_3 - 2J_1P_2 + W_1 \equiv 0,$ *or*
$P_1P_2 \not\equiv 0,\ p_{27} \equiv 0,\ 3P_1P_3 - 2J_1P_2 \equiv 0,$ *or*
50 $P_2P_5 \not\equiv 0,\ P_1 \equiv J_2P_5 - J_4P_2 + W_1 + W_2 \equiv 0,$ *or*
50 $P_2 \equiv 0,\ P_1 \not\equiv 0,\ J_1 + p_{27} \not\equiv 0,\ J_1p_{27} \equiv 0;$

2 for $P_2 \not\equiv 0,\ P_1 \equiv P_5 \equiv J_2P_5 - J_4P_2 + W_1 + W_2 \equiv 0,$ *or*
$P_2 \equiv p_{27} \equiv 0,\ P_1 \not\equiv 0,\ J_1 = 0,$ *or*
$P_1 \equiv P_2 \equiv 0,\ p_{27} \not\equiv 0;$

0 for $P_1 \equiv P_2 \equiv p_{27} \equiv 0,$

where P_1, P_2, P_3, P_5, p_{27}, J_1, J_2, J_4 are taken from (5), W_1 is taken from (22), and W_2 - from (34).

Let introduce the following designations:

$$M_1 = M_1(P_1P_2 \not\equiv 0,\ 3P_1P_3 - 2J_1P_2 + W_1 \not\equiv 0);$$

$$M_2 = M_2(p_{27} \equiv 0,\ P_1P_2(3P_1P_3 - 2J_1P_2) \not\equiv 0);$$

$$M_3 = M_3(p_{27} \equiv 0,\ P_1 \equiv 0,\ P_2(J_2P_5 - J_4P_2) \not\equiv 0);$$
$$M_4 = M_4(P_1 \equiv 0,\ P_2 \not\equiv 0,\ J_2P_5 - J_4P_2 + W_1 + W_2 \not\equiv 0);$$
$$M_5 = M_5(P_2 \equiv 0,\ J_1p_{27} \not\equiv 0);$$
$$M_6 = M_6(P_1P_2 \not\equiv 0,\ 3P_1P_3 - 2J_1P_2 + W_1 \equiv 0);$$
$$M_7 = M_7(P_1P_2 \not\equiv 0,\ p_{27} \equiv 0,\ 3P_1P_3 - 2J_1P_2 \equiv 0);$$
$$M_8 = M_8(P_2P_5 \not\equiv 0,\ P_1 \equiv J_2P_5 - J_4P_2 + W_1 + W_2 \equiv 0);$$
$$M_9 = M_9(P_2 \equiv 0,\ P_1 \not\equiv 0,\ J_1 + p_{27} \not\equiv 0,\ J_1p_{27} \equiv 0);$$
$$M_{10} = M_{10}(P_2 \not\equiv 0,\ P_1 \equiv P_5 \equiv J_2P_5 - J_4P_2 + W_1 + W_2 \equiv 0);$$
$$M_{11} = M_{11}(P_2 \equiv p_{27} \equiv 0,\ P_1 \not\equiv 0,\ J_1 = 0);$$
$$M_{12} = M_{12}(P_1 \equiv P_2 \equiv 0,\ p_{27} \not\equiv 0);$$
$$M_{13} = M_{13}(P_1 \equiv P_2 \equiv p_{27} \equiv 0). \tag{51}$$

According to Definitions 1 and 2 from Theorem 1 follows

Theorem 2. *Sets M_i $(1 \leq i \leq 13)$ from (51) form $GL(2,R)$-invariant division of the set $E(A)$ of the coefficient of the system (1), i.a.*

$$\bigcup_{i=1}^{13} M_i = E(A),\ M_i \cap_{i \neq j} M_j = \oslash,$$

where each M_i is $GL(2,R)$-invariant.

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