BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(43), 2003, Pages 53–57
ISSN 1024–7696

On a nonlinear differential subordination I

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Abstract. We find conditions on the complex-valued functions A, B, C, D in the unit disc U such that the differential inequality

$$|A(z)z^2p''(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

implies $|p(z)| < N$, where p is analytic in U, with $p(0) = 0$.

Mathematics subject classification: 30C80.
Keywords and phrases: Differential subordination, dominant.

1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots, \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In [1] chapter IV, the authors have analyzed a first-order linear differential subordination

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \tag{1}$$

where A, B, C, D and h are complex-valued functions in the unit disc, where $p \in \mathcal{H}[0, n]$. A more general version of (1) is given by:

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega,$$

where $\Omega \subset \mathbb{C}$.

In [2] we found conditions on the complex-valued functions A, B, C, D in the unit disc U and the positive numbers M and N such that

$$|A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

implies $|p(z)| < N$, where $p \in \mathcal{H}[0, n]$.

In this paper we shall consider the following particular second-order nonlinear differential subordination given by the inequality

$$|A(z)z^2p''(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M, \tag{2}$$

where $p \in \mathcal{H}[0, n]$.

We find conditions on complex-valued functions A, B, C, D and the positive numbers M and N such that (2) implies $|p(z)| < N$, where $p \in \mathcal{H}[0, n]$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3h [1, p. 34].

Lemma A. [1, p. 34] *Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $M > 0$, $N > 0$ satisfy*

$$|\psi(Ne^{i\theta}, L; z)| \geq M \tag{3}$$

whenever $\mathrm{Re}\ [Le^{-i\theta}] \geq n(n-1)M$, $z \in U$ *and* $\theta \in \mathbb{R}$, *where n is a positive integer.*

If $p \in \mathcal{H}[0, n]$ and $|\psi(p(z), z^2p''(z); z)| < M$ then $|p(z)| < N$.

2 Main results

Theorem. *Let $M > 0$, $N > 0$, and let n be a positive integer. Suppose that the functions $A, B, C, D : U \to \mathbb{C}$ satisfy $A(z) \neq 0$,*

$$\mathrm{Re}\ \frac{C(z)}{A(z)} \geq \frac{M + N^2|B(z)| + D(z)|}{N|A(z)|}. \tag{4}$$

If $p \in \mathcal{H}[0, n]$ and

$$|A(z)z^2p''(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

then $|p(z)| < N$.

Proof. Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be defined by

$$\psi(p(z), z^2p''(z); z) = A(z)z^2p''(z) + B(z)p^2(z) + C(z)p(z) + D(z). \tag{5}$$

From (2) we have

$$|\psi(p(z), z^2p''(z); z)| < M, \text{ for } z \in U. \tag{6}$$

Using (4) and (5) we have

$$|\psi(Ne^{i\theta}, L, z)| = |A(z)L + B(z)N^2e^{2i\theta} + C(z)Ne^{i\theta} + D(z)| =$$

$$= |A(z)Le^{-i\theta} + B(z)N^2e^{i\theta} + C(z)N + D(z)e^{-i\theta}| =$$

$$= |A(z)| \left| Le^{-i\theta} + \frac{B(z)}{A(z)}N^2e^{i\theta} + \frac{C(z)}{A(z)}N + \frac{D(z)}{A(z)}e^{-i\theta} \right| \geq$$

$$\geq |A(z)| \left[\left| Le^{-i\theta} + \frac{B(z)}{A(z)} N^2 e^{i\theta} + \frac{C(z)}{A(z)} N \right| - \left| \frac{D(z)}{A(z)} \right| \right] \geq$$

$$\geq |A(z)| \left[\left| Le^{-i\theta} + \frac{C(z)}{A(z)} N \right| - N^2 \left| \frac{B(z)}{A(z)} \right| - \left| \frac{D(z)}{A(z)} \right| \right] \geq$$

$$\geq |A(z)| \left[\text{Re } Le^{-i\theta} + N \text{Re } \frac{C(z)}{A(z)} - N^2 \left| \frac{B(z)}{A(z)} \right| - \left| \frac{D(z)}{A(z)} \right| \right] \geq$$

$$\geq |A(z)| \left[n(n-1)M + N \text{Re } \frac{C(z)}{A(z)} - N^2 \left| \frac{B(z)}{A(z)} \right| - \left| \frac{D(z)}{A(z)} \right| \right] \geq$$

$$\geq |A(z)| \left[N \text{Re } \frac{C(z)}{A(z)} - N^2 \left| \frac{B(z)}{A(z)} \right| - \left| \frac{D(z)}{A(z)} \right| \right] \geq M.$$

Hence condition (3) holds and by Lemma A we deduce that (6) implies $|p(z)| < N$. □

Instead of prescribing the constant N in Theorem, in some cases we can use (4) to determine an appropriate $N = N(M, n, A, B, C, D)$ so that (2) implies $|p(z)| < N$.

This can be accomplished by solving (4) for N and by taking the supremum of the resulting function over U.

Condition (4) is equivalent to:

$$N^2 |B(z)| - N|A(z)| \text{Re } \frac{C(z)}{A(z)} + M + |D(z)| \leq 0. \tag{7}$$

If we suppose $B(z) \neq 0$, then the inequality (7) holds if

$$|A(z)| \text{Re } \frac{C(z)}{A(z)} \geq 2\sqrt{|B(z)|(M + |D(z)|)} \tag{8}$$

If (8) holds, the roots of the trinomial in (7) are

$$N_{1,2} = \frac{|A(z)| \text{Re } \frac{C(z)}{A(z)} \pm \sqrt{\left[|A(z) \text{Re } \frac{C(z)}{A(z)} \right]^2 - 4M|B(z)|(M + |D(z)|)}}{2|B(z)|}.$$

We let

$$N = \frac{2(M + |D(z)|)}{|A(z)| \text{Re } \frac{C(z)}{A(z)} + \sqrt{\left[|A(z)| \text{Re } \frac{C(z)}{A(z)} \right]^2 - 4M|C(z)|(M + |D(z)|)}}.$$

If this supremum is finite, the Theorem can be rewritten as follows:

Corollary 1. *Let $M > 0$ and let n be a positive integer. Suppose that $p \in \mathcal{H}[0, n]$ and let the functions $A, B, C, D : U \to \mathbb{C}$, with $A(z) \neq 0$.*

If

$$N=\sup_{|z|<1}\frac{2(M+|D(z)|)}{|A(z)|\mathrm{Re}\ \dfrac{C(z)}{A(z)}+\sqrt{\left[|A(z)|\mathrm{Re}\ \dfrac{C(z)}{A(z)}\right]^2-4M|C(z)|(M+|D(z)|)}}<\infty$$

then

$$|A(z)z^2p''(z)+B(z)p^2(z)+C(z)p(z)+D(z)|<M$$

implies $|p(z)|<N$.

Let $n=1$, $A(z)=-4$, $B(z)=12-5i$, $C(z)=-20+7i$, $D(z)=1-\sqrt{3}i$, $M=4$, we find $N=\dfrac{6}{10+\sqrt{22}}$.

In this case from Corollary 1 we deduce

Example 1. If $p\in\mathcal{H}[0,1]$, then

$$|-4z^2p''(z)+(12-5i)p^2(z)+(-20+7i)p(z)+(1-\sqrt{3}i)|<4$$

implies

$$|p(z)|<\frac{6}{10+\sqrt{22}}.$$

If $n=2$, $A(z)=6$, $B(z)=4+3i$, $C(z)=18-5i$, $D(z)=2\sqrt{3}+2i$, $M=5$, we find $N=\dfrac{6}{6+\sqrt{26}}$.

In this case from Corollary 1 we deduce

Example 2. If $p\in\mathcal{H}[0,2]$ then

$$|6z^2p''(z)+(4+3i)p^2(z)+(18-5i)p(z)+(2\sqrt{3}+2i)|<5$$

implies

$$|p(z)|<\frac{6}{10+\sqrt{22}}.$$

If $A(z)=A>0$ then the Theorem can be rewritten as follows:

Corollary 2. *Let $M>0$, $N>0$ and let n be a positive integer. Suppose that the functions $B,C,D:U\to\mathbb{C}$ satisfy*

$$\mathrm{Re}\ C(z)\geq\frac{M+N^2|B(z)|+|D(z)|}{N}.$$

If $p\in\mathcal{H}[0,n]$ and

$$|Az^2p''(z)+B(z)p^2(z)+C(z)p(z)+D(z)|<M$$

then $|p(z)|<N$. $\square$

If $n=1$, $A=8$, $B(z)=\sqrt{3}+i$, $C(z)=20-5i$, $D(z)=3-4i$, $M=8$, $N=2$.

In this case from Corollary 2 we deduce:

Example 3. If $p \in \mathcal{H}[0,1]$, and

$$|8z^2p''(z) + (\sqrt{3}+i)p^2(z) + (20-5i)p(z) + (3-4i)| < 8$$

then $|p(z)| < 2$.

If $n=2$, $A=4$, $B(z) = -1-i\sqrt{3}$, $C(z) = 16+4i$, $D(z) = 2+3i$, $M=4$, $N=1$.

In this case from Corollary 2 we deduce:

Example 4. If $p \in \mathcal{H}[0,1]$, and

$$|4z^2p''(z) + (-1-i\sqrt{3})p^2(z) + (16+4i)p(z) + (2+3i)| < 4$$

then $|p(z)| < 1$.

References

[1] Miller S.S., Mocanu P.T., *Differential Subordinations. Theory and Applications.* – Marcel Dekker Inc., New York, Basel, 2000.

[2] Oros Gh., Oros G.I., *On a first-order nonlinear differential subordination I.* – (submitted)

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Received April 10, 2003