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Weak convergence of the distributions of Markovian random evolutions in two and three dimensions

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Abstract. We consider Markovian random evolutions performed by a particle moving in R^2 and R^3 with some finite constant speed v randomly changing its directions at Poisson-paced time instants of intensity $\lambda > 0$ uniformly on the S_2 and S_3-spheres, respectively. We prove that under the Kac condition

$$v \to \infty, \qquad \lambda \to \infty, \qquad \frac{v^2}{\lambda} \to c, \qquad c > 0$$

the transition laws of the motions weakly converge in an appropriate Banach space to the transition law of the two- and three-dimensional Wiener process, respectively, with explicitly given generators.

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1 Introduction

The processes of random evolution in some phase space are being described by the equality

$$\frac{\partial f}{\partial t} = \mathcal{A} f + \Lambda f, \tag{1}$$

which, in a certain sense, can be referred to as Kolmogorov equation for the random evolution. In this equation (1) $\mathcal{A}$ is some purely spacial operator of a special form acting in an appropriate Banach space, namely, some sort of diagonal matrix differential operator acting in the space of sufficiently smooth functions, and Λ is the infinitesimal operator of a stochastic process governing the evolution. The particular form of equation (1) is determined by the type of the evolution space and kind of the controlling stochastic process. For instance, if the evolution is driven by a continuous-time Markov chain with a finite number of states $n, n \geq 2$, then equation (1) takes the form of a system of n first-order PDEs, $\mathcal{A}$ is some diagonal $(n \times n)$-matrix differential operator acting in the space of differentiable vector-functions and Λ is a scalar infinitesimal $(n \times n)$-matrix of the embedded Markov chain.

The operator $\mathcal{A}$ is responsible for the propagation velocity of the evolution and Λ deals with the intensity of the switching stochastic process. Therefore, it is natural to represent these operators in the form $\mathcal{A} = \varepsilon_1 A,\ \Lambda = \varepsilon_2 Q$, where the parameters

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ε_1 and ε_2 have the sense of the velocity of the evolution and the intensity of the governing stochastic process, respectively, and the operators A and Q do not depend on ε_1 and ε_2.

The systems of the form

$$\frac{\partial f}{\partial t} = \varepsilon_1 A f + \varepsilon_2 Q f \tag{2}$$

have become the subject of a great deal of researches, among which the problem of diffusion approximation of random evolutions was of a special interest. It is clear that in order the evolution to have a diffusion limit, its velocity and rate of switches must satisfy some sort of equilibrium condition. In other words, the parameters ε_1 and ε_2 in (2) must be connected between themselves by that or another relationship.

In the case a random evolution is controlled by a continuous-time homogeneous Markov chain with n states, the limit behaviour of a process governed by the system (2) with $\varepsilon_1 = \varepsilon,\ \varepsilon_2 = \varepsilon^2$, A is some diagonal $(n \times n)$-matrix differential operator, Q is a $(n \times n)$-matrix of infinitesimal parameters, has been examined by Pinsky [12], Griego and Hersh [1], Hersh and Papanicolaou [3], who have given the diffusion approximation theorems as $\varepsilon \to \infty$. A system of the form (2) has thoroughly been studied by Hersh and Pinsky [4] and a limit theorem has been given as the ratio $(\varepsilon_1/\varepsilon_2) \to 0$. An abstract version of these diffusion approximation theorems has been given by Kurtz [9] for arbitrary evolution space and kind of the controlling Markov process. The reader interested in more details on the subject should address to the survey article by Hersh [2] and, especially, to the monographs by Pinsky [14] and by Korolyuk and Swishchuk [8].

The most interesting case of random evolution performed by a particle moving in $R^m, m \geq 1$, at some finite constant speed v subject to the control of a homogeneous Poisson process of rate $\lambda > 0$ (so-called transport process), is being described by an equation of the form (2) with $\varepsilon_1 = v$ and $\varepsilon_2 = \lambda$. It is known that in many such cases the limiting diffusion process arises if v and λ satisfy the following Kac condition

$$v \to \infty, \qquad \lambda \to \infty, \qquad \frac{v^2}{\lambda} \to c, \qquad c > 0 \tag{3}$$

and the transition functions of the evolution (as a two-parameter family of distributions depending on v and λ) weakly converge to the transition function of a corresponding Brownian motion. Moreover, in all the cases (a very few ones) when the transition laws were obtained in an explicit form (see, for instance, Orsingher [10], theorem 1, for the transition law of the Goldstein-Kac telegraph process in R^1, and Orsingher [11], theorem 3.1, for the transition law of a random evolution with four directions in R^2), the condition (3) provided the pointwise convergence of the transition functions of the motion to the transition function of the Wiener process.

A Markovian random evolution with an arbitrary number of directions $n, n \geq 2$, in R^2 has been studied by Kolesnik and Turbin [7], and a nth order hyperbolic equation with constant coefficients governing the transition law of the motion has been obtained. It was also shown that under the Kac condition (3) the governing hyperbolic operator turns into the classical parabolic diffusion operator in R^2 with the generator

$$G_n = \frac{c(n-1)}{2n}\Delta, \qquad n \geq 3 \tag{4}$$

where Δ is the two-dimensional Laplacian. A diffusion approximation theorem proved in Kolesnik [6] also stated that under the Kac condition (3) the transition laws of the evolution weakly converge in a suitably chosen Banach space to the transition law of the Wiener process in R^2 with generator (4).

One should note that in both these works it was supposed that under each change of direction the particle took on any new one uniformly with probability $1/(n-1)$, that is, it could not preserve its current direction. However, if we suppose that every new direction can be taken on uniformly with equal probabilities $1/n$ (i.e. the transition probabilities of the embedded Markov chain are $p_{ij} = 1/n$ for any i and j), then replacing everywhere $1/(n-1)$ for $1/n$ we obtain that the generator of the limiting Wiener process for any $n \geq 3$ is

$$G_n = \frac{c}{2}\Delta, \tag{5}$$

and there is not the number of directions n in the right-hand side of (5). In other words, under the full symmetry of the motion the limiting Wiener process *does not depend* on the number of directions n. It is worth to note that generator (5) also arises from (4) as $n \to \infty$.

This amazing fact allows us to expect that for an evolution with the continuum number of directions (i.e. when the particle chooses new directions uniformly on the unit circumference) the limiting Wiener process will have the same generator (5). Proof of this statement is one of the principal results of our paper.

Studying of random evolutions with the continuum number of directions in $R^m, m \geq 2$, is an extremely interesting, natural and practically useful problem. Although the equation governing such a motion is not obtained yet, nevertheless we are able to present the results concerning limiting behaviour of the transition laws of the evolutions in R^2 and R^3 under the Kac condition (3).

The main tool of our research is a diffusion approximation method given in Kurtz [9]. In Section 2, for the reader's convenience, we shall briefly remind the main points of this method in a form convenient for further applications. In Section 3 we shall apply it to the problem of studying the behaviour of the transition laws of a random evolution in R^2 governed by a jump Markov process on the S_2-sphere (unit circumference). We will show that under the Kac condition (3) the transition laws of the evolution weakly converge in an appropriate Banach space to the transition law of the two-dimensional Brownian motion with zero drift and the variance $\sqrt{c}$. In Section 4 we will give a similar result for a random evolution in R^3 driven by a jump Markov process on the S_3-sphere (surface of the unit 3D-ball) and will prove the weak convergence of the transition laws of the motion to the transition law of the Wiener process in R^3 with zero drift and the variance $\sqrt{2c/3}$.

2 Kurtz's Approximation Method

Let $U(t)$ and $S(t)$ be strongly continuous semigroups of linear contractions on a Banach space L with infinitesimal operators A and B, respectively. Let $\mathcal{D}(A)$ and $\mathcal{D}(B)$ be the domains of A and B. Assume that for each sufficiently large α, the closure of $A + \alpha B$ is the infinitesimal operator of a strongly continuous semigroup $T_\alpha(t)$ on L. Also suppose that B is the closure of B restricted to $\mathcal{D}(A) \cap \mathcal{D}(B)$. We are interested in the behaviour of $T_\alpha(t)$ as α goes to infinity.

Define the operator P on L by the equality

$$Pf = \lim_{\gamma \to 0} \gamma \int_0^\infty e^{-\gamma t} S(t) f \, dt, \tag{6}$$

and suppose that the limit in the right-hand side of (6) exists for every $f \in L$. It is known (see Hille and Phillips [5], page 516) that operator P defined by (6) is a bounded linear projection, i.e. $P^2 = P$.

Denote by $\mathcal{R}(P)$ the image of the operator P. Let

$$D = \{f \in \mathcal{R}(P) \; : \; f \in \mathcal{D}(A)\}$$

and for $f \in D$ define the operator C by the equality $Cf = PAf$. Kurtz's approximation method is given by the following theorem.

Theorem [Kurtz [9], theorem 2.2]. *Let $U(t),\ S(t),\ T_\alpha(t),\ D,\ C$ be defined as above. Suppose that for all $f \in D$*

$$Cf = 0. \tag{7}$$

Let

$$D_0 = \{f \in D \; : \; \exists\, h \in \mathcal{D}(A) \cap \mathcal{D}(B) \;\text{ such that }\; Bh = -Af\}. \tag{8}$$

For $f \in D_0$ define the operator C_0 by the equality

$$C_0 f = PAh. \tag{9}$$

Suppose that

$$\overline{\mathcal{R}(\mu - C_0)} \supset \overline{D_0} \tag{10}$$

for some $\mu > 0$.

Then the closure of C_0 restricted so that $C_0 f \in \overline{D_0}$ is the infinitesimal operator of a strongly continuous contraction semigroup $T(t)$ defined on $\overline{D_0}$ and for all $f \in \overline{D_0}$

$$T(t)f = \lim_{\alpha \to \infty} T_\alpha(\alpha t) f.$$

This theorem gives an effective method of obtaining approximation results for a wide class of stochastic processes. First of all one should note that the conditions of the theorem are not too burdensome. The equality (7) is some sort of symmetry condition which in practice can often be provided (if needed) by simple transformations.

Fulfilment of the condition (10) can be provided by the choice of an appropriate Banach space. Therefore, there are two crucial points in this method. The first one concerns finding of a solution $h \in \mathcal{D}(A) \cap \mathcal{D}(B)$ of the equation

$$Bh = -Af \tag{11}$$

for every element $f \in D_0$. The second point concerns the possibility of computing the projector P defined by (6). The main question here is the existence of the limit in the right-hand side of (6).

In the case of Markovian random evolutions the projector P can be found by means of a more explicit formula. Let $V(t)$ be a temporally homogeneous Markov process with measurable state space $(E,\ \mathcal{E})$ and transition function $P(t, x, \Gamma)$. Then the semigroup $S(t)$ in the Banach space of bounded strongly measurable functions $f\ :\ E \to L$ with the *sup*-norm is defined by

$$S(t)f(x) = \int_E f(y)\ P(t, x, dy),$$

and the projector P is explicitly given by the formula

$$Pf(x) = \int_E f(y)\ P(x, dy), \tag{12}$$

where $P(x, \Gamma)$ is the limiting distribution, assumed to exist, of the process $V(t)$ starting from x, or the weak limit as $t \to \infty$ of the transition function $P(t, x, \Gamma)$. One should note that formula (12) takes an especially simple form if the limiting distribution $P(x, \Gamma)$ is uniform.

If h and P are found and the conditions (7) and (10) are fulfilled then, according to the conclusion of the Kurtz's theorem, one can assert that the transition laws of the random evolution weakly converge to the transition law of a process with generator given by the closure of C_0.

In the next sections we will apply this method to the Markovian random evolutions in R^2 and R^3 and prove that their transition functions weakly converge to the transition function of the two- and three-dimensional Brownian motion, respectively, with explicitly given generators.

3 Diffusion Approximation Theorem in R^2

Consider the following planar stochastic motion. A particle starts at the moment $t = 0$ from the origin $x = y = 0$ of the plane R^2 taking initial random direction uniformly on the S_2-sphere (unit circumference) and moves with some constant finite speed v. At every time instant $t > 0$ it can have some random direction of motion $E_\varphi,\ \varphi \in [0, 2\pi)$ which forms the angle φ with x-axis. In other words, the direction E_φ is oriented like the vector $e_\varphi = (\cos\varphi,\ \sin\varphi),\ \varphi \in [0, 2\pi)$. The motion is controlled by a homogeneous Poisson process of rate $\lambda > 0$ as follows. When a Poisson event occurs, the particle instantly takes on a new random direction distributed uniformly on S_2 and continues its motion in the chosen direction with the same speed v until the next Poisson event occurs, then it takes on a new random direction again, and so on. Thus, the evolution is controlled by the jump Markov process Φ_t on the unit circumference S_2.

Let $\Xi(t) = (X_t, Y_t)$ denote the particle's position in the plane at some instant $t > 0$. Since the motion depends on v and λ then, in fact, we deal with a two-parameter family of stochastic processes $\Xi_v^\lambda(t)$. Bearing this in mind, we omit these indices in the sequel.

The main goal of this section is to study the behaviour of the transition laws of $\Xi(t)$ as the intensity of transitions λ tends to infinity and, according to λ, the particle speed v increases as well. The accordance between the growth rates of λ and v is determined by the Kac condition (3).

Since the sample paths of $\Xi(t)$ are continuous and differentiable almost everywhere and the velocity of the process is finite, the distribution of $\Xi(t)$ consists of the absolutely continuous component concentrated strictly inside the circle

$$K_t = \{(x,y) \in R^2 : x^2 + y^2 < v^2t^2\}, \quad t > 0$$

and the singular component on the boundary

$$B_t = \{(x,y) \in R^2 : x^2 + y^2 = v^2t^2\}, \quad t > 0.$$

Therefore there exist the partial (with respect to directions) transition densities $f_\varphi = f_\varphi(x,y,t), \ (x,y) \in K_t, \ t > 0, \ \varphi \in [0, 2\pi)$ of the absolutely continuous component of $\Xi(t)$ defined by the equality

$$f_\varphi(x,y,t)dxdyd\varphi = Prob\{x \le X_t < x + dx, \ y \le Y_t < y + dy, \ \varphi \le \Phi_t < \varphi + d\varphi\}$$

Kolmogorov equation (1) written down for these transition densities has the form of the integro-differential equation

$$\frac{\partial f_\varphi}{\partial t} = -v\cos\varphi \frac{\partial f_\varphi}{\partial x} - v\sin\varphi \frac{\partial f_\varphi}{\partial y} - \lambda f_\varphi + \frac{\lambda}{2\pi}\int_0^{2\pi} f_\theta d\theta, \qquad \varphi \in [0, 2\pi). \qquad (13)$$

Equation (13) is a particular case (for the uniform dissipation function identically equal to $1/(2\pi)$) of a some more general equation with an arbitrary dissipation function given in the monograph by Tolubinsky [15], page 40. One should note that the integral term in (13) appears due to the continuum number of directions. This is the main difference of the motion from the model with a finite number of directions studied in Kolesnik and Turbin [7] and Kolesnik [6] where only PDEs arose.

Consider the Banach space $\mathcal{B}$ of twice continuously differentiable functions on $R^2 \times (0, \infty)$ vanishing at infinity. The transition densities f_φ can be considered as the one-parameter family of functions $f = \{f_\varphi, \ \varphi \in [0, 2\pi)\}$ belonging to $\mathcal{B}$.

Introduce the one-parameter family $\mathcal{A} = \{A_\theta, \ \theta \in [0, 2\pi)\}$ of operators acting in $\mathcal{B}$ where

$$A_\theta = -v\cos\theta \frac{\partial}{\partial x} - v\sin\theta \frac{\partial}{\partial y}.$$

Define the action of $\mathcal{A}$ on f as

$$\mathcal{A}f = \{\delta(\theta, \varphi) A_\theta f_\varphi, \quad \theta, \varphi \in [0, 2\pi)\} \tag{14}$$

where

$$\delta(\theta, \varphi) = \begin{cases} 1, & \text{if } \theta = \varphi \\ 0, & \text{otherwise} \end{cases}$$

is the generalized Kronecker delta-symbol of rank 2. The operator $\mathcal{A}$ in (14) is an analogue of a diagonal matrix differential operator and the family f is the continuum analogue of the vector-function of partial transition densities appearing in the finite-state case (see Kolesnik [6], formula (2)).

Introduce now the operator Λ acting on f by the following formula

$$\Lambda f = -\lambda f + \frac{\lambda}{2\pi} \int_0^{2\pi} f_\theta d\theta. \tag{15}$$

Then equality (13) can be rewritten as follows

$$\frac{\partial f}{\partial t} = \mathcal{A}f + \Lambda f \tag{16}$$

and it has exactly the form of equation (1).

The principal result of this section is given by the following theorem.

2D-Diffusion Approximation Theorem. *Let the Kac condition (3) be fulfilled. Then in the Banach space $\mathcal{B}$ the semigroups generated by the transition functions of the process $\Xi(t)$ converge to the semigroup generated by the transition function of the Wiener process in R^2 with generator*

$$G = \frac{c}{2}\Delta \tag{17}$$

where Δ is the two-dimensional Laplace operator.

Remark. Note that generator (17) also formally appears from formula (13) of Kolesnik [6] and formula (4.3) of Kolesnik and Turbin [7] as $n \to \infty$.

Remark. One should also note that for the particular case when the limiting constant $c = 1$, the generator (17) coincides with the evolutionary operator given in Proposition 4.8 of the paper by Pinsky [13] for the dimension $m = 2$.

Proof. According to formulas (8) and (11) of the Kurtz's theorem above, we need to find a solution h of the equation

$$\Lambda h = -\mathcal{A}f \tag{18}$$

for arbitrary function (family) $f \in D_0$. As is easy to see, such a solution for any differentiable function f is given by the formula

$$h = \frac{1}{\lambda}\mathcal{A}f + \frac{1}{2\pi} \int_0^{2\pi} f_\theta d\theta. \tag{19}$$

Really, taking into account that for any $f \in \mathcal{B}$

$$\int_0^{2\pi} \mathcal{A} f \, d\theta = \left(\int_0^{2\pi} A_\theta \, d\theta \right) \, f = 0 \tag{20}$$

and using (15) and (19) we obtain

$$\Lambda h = -\lambda \left(\frac{1}{\lambda} \mathcal{A} f + \frac{1}{2\pi} \int_0^{2\pi} f_\varphi d\varphi \right) + \frac{\lambda}{2\pi} \int_0^{2\pi} \left(\frac{1}{\lambda} \mathcal{A} f + \frac{1}{2\pi} \int_0^{2\pi} f_\varphi d\varphi \right) d\theta = -\mathcal{A} f$$

and equality (18) is fulfilled.

Our next step is to compute the projector P given by formula (12). Since the limiting distribution of the governing Markov process on S_2 is uniform with the density $1/(2\pi)$ then formula (12) simplifies, and the projector is given by

$$Pf = \frac{1}{2\pi} \int_0^{2\pi} f_\varphi d\varphi. \tag{21}$$

Then, according to (9), (19) and (21), we obtain

$$C_0 f = P \mathcal{A} h = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{\lambda} \mathcal{A}^2 f + \frac{1}{2\pi} \mathcal{A} \int_0^{2\pi} f_\varphi d\varphi \right) d\theta.$$

The well-known equalities

$$\int_0^{2\pi} \sin^2 \varphi \, d\varphi = \pi, \qquad \int_0^{2\pi} \cos^2 \varphi \, d\varphi = \pi, \qquad \int_0^{2\pi} \sin \varphi \cos \varphi \, d\varphi = 0$$

yield the formula

$$\int_0^{2\pi} \mathcal{A}^2 f \, d\theta = \left(\int_0^{2\pi} A_\theta^2 \, d\theta \right) f = \pi v^2 \Delta f \tag{22}$$

where Δ is the two-dimensional Laplacian and therefore, taking into account (20) and (22), for any $f \in \mathcal{B}$ we have

$$C_0 f = \left(\frac{1}{2\pi\lambda} \int_0^{2\pi} A_\theta^2 d\theta \right) f + \frac{1}{4\pi^2} \left(\int_0^{2\pi} A_\theta \, d\theta \right) \left(\int_0^{2\pi} f_\varphi d\varphi \right) = \frac{v^2}{2\lambda} \Delta f.$$

Thus, we obtain

$$C_0 = \frac{v^2}{2\lambda} \Delta$$

and therefore generator (17) under the Kac condition (3) is the limiting operator of the evolution.

It remains to check conditions (7) and (10) of the Kurtz's theorem. Taking into account equality (20) for any f continuously differentiable we have

$$Cf = P \mathcal{A} f = \frac{1}{2\pi} \left(\int_0^{2\pi} A_\theta \, d\theta \right) f = 0$$

and thus condition (7) is fulfilled.

In order to check condition (10) it is sufficient to show that for any function f twice continuously differentiable there exists a solution g of the equation

$$(\mu - C_0)g = f \tag{23}$$

for some $\mu > 0$. One can easily see that for any $\mu > 0$ equation (23) takes the form of an inhomogeneous Klein-Gordon equation (or Helmholtz equation with a purely imaginary constant) with a sufficiently smooth right-hand part, and existence of its solution is well-known from the general PDEs theory. Thus, condition (10) is also fulfilled.

Therefore, by the Kurtz's approximation theorem, one can assert that under the Kac condition (3) the semigroups generated by the transition laws of the process $\Xi(t)$ converge in $\mathcal{B}$ to the semigroup generated by the transition law of the Wiener process in R^2 with generator (17). □

4 Diffusion Approximation Theorem in R^3

In this section we give a similar result concerning 3-dimensional random evolution. A particle starts at the moment $t = 0$ from the origin $x = y = z = 0$ of the space R^3 taking initial random direction uniformly on the S_3-sphere (surface of the unit 3D-ball) and moves with some constant finite speed v. At every time instant $t > 0$ it can have some random direction of motion $\omega \in S_3$ where ω is a spacial (bodial) angle. The motion is driven by a Poisson process of rate $\lambda > 0$ as follows. When a Poisson event occurs, the particle instantly takes on a new random direction distributed uniformly on S_3 and continues its motion in the chosen direction with the same speed v until the next Poisson event occurs, then it takes on a new random direction again, and so on. Thus, the evolution is controlled by the jump Markov process Φ_t on S_3.

Let $\Xi(t) = (X_t, Y_t, Z_t)$ denote the particle's position in the space R^3 at some instant $t > 0$. The main goal of this section is to study the behaviour of the transition laws of $\Xi(t)$ under the Kac condition (3).

Like in the planar case, the distribution of $\Xi(t)$ consists of the absolutely continuous component concentrated strictly inside the ball

$$K_t = \{(x, y, z) \in R^3 : x^2 + y^2 + z^2 < v^2t^2\}, \quad t > 0$$

and the singular component on the boundary

$$B_t = \{(x, y, z) \in R^3 : x^2 + y^2 + z^2 = v^2t^2\}, \quad t > 0$$

Therefore there exist the partial (with respect to directions) transition densities $f_\omega = f_\omega(x, y, z, t), \ (x, y, z) \in K_t, \ \omega \in S_3, \ t > 0$, of the absolutely continuous component of $\boldsymbol{\Xi}(t)$ defined by the equality

$$f_\omega(x,y,z,t)\;dx\;dy\;dz\;\mu(d\omega) =$$
$$Prob\{x \le X_t < x+dx,\; y \le Y_t < y+dy,\; z \le Z_t < z+dz,\; \Phi_t \in d\omega\}$$

where $\mu(d\omega)$ is the measure of the elementary spacial angle $d\omega$.

Since in R^3 any direction ω is determined by the ordered pair of two planar angles (φ,ψ), $\varphi \in [0,2\pi)$, $\psi \in [0,\pi)$, and the measure of the elementary spacial angle $d\omega$ is equal to

$$\mu(d\omega) = \sin\psi\; d\psi\; d\varphi \tag{24}$$

then Kolmogorov equation (1) written down for the transition densities $f_\omega = f_{\varphi,\psi}$ has the form of the integro-differential equation

$$\frac{\partial f_{\varphi,\psi}}{\partial t} = -v\sin\psi\cos\varphi\frac{\partial f_{\varphi,\psi}}{\partial x} - v\sin\psi\sin\varphi\frac{\partial f_{\varphi,\psi}}{\partial y} - v\cos\psi\frac{\partial f_{\varphi,\psi}}{\partial z}$$
$$-\lambda f_{\varphi,\psi} + \frac{\lambda}{4\pi}\int_{S_3} f_\omega\;\mu(d\omega), \qquad \varphi \in [0,2\pi),\; \psi \in [0,\pi). \tag{25}$$

Equation (25) is a particular case (for the uniform dissipation function identically equal to $1/(4\pi)$) of a some more general equation with an arbitrary dissipation function given in the monograph by Tolubinsky [15], page 40.

Consider the Banach space $\mathcal{B}$ of twice continuously differentiable functions on $R^3 \times (0,\infty)$ vanishing at infinity. The transition densities $f_{\varphi,\psi}$ can be considered as the two-parameter family of functions $f = \{f_{\varphi,\psi},\; \varphi \in [0,2\pi),\; \psi \in [0,\pi)\}$ belonging to $\mathcal{B}$.

Introduce the two-parameter family $\mathcal{A} = \{A_{\theta,\nu},\;\; \theta \in [0,2\pi),\; \nu \in [0,\pi)\}$ of operators acting in $\mathcal{B}$ where

$$A_{\theta,\nu} = -v\sin\psi\cos\varphi\frac{\partial}{\partial x} - v\sin\psi\sin\varphi\frac{\partial}{\partial y} - v\cos\psi\frac{\partial}{\partial z}.$$

Define the action of $\mathcal{A}$ on f as

$$\mathcal{A}f = \{\delta(\theta,\varphi)\delta(\nu,\psi)A_{\theta,\nu}f_{\varphi,\psi},\;\; \theta,\varphi \in [0,2\pi),\; \nu,\psi \in [0,\pi)\} \tag{26}$$

where $\delta(\cdot,\cdot)$ is the generalized Kronecker delta-symbol of rank 2 defined above.

Introduce now the operator Λ acting on f in the following way

$$\Lambda f = -\lambda f + \frac{\lambda}{4\pi}\int_{S_3} f_\omega\;\mu(d\omega). \tag{27}$$

Then equality (25) can be rewritten as

$$\frac{\partial f}{\partial t} = \mathcal{A}f + \Lambda f$$

having the form of equation (1) and similar to (16).

The principal result of this section is given by the following theorem.

3D-Diffusion Approximation Theorem. *Let the Kac condition (3) be fulfilled. Then in the Banach space $\mathcal{B}$ the semigroups generated by the transition functions of the process $\Xi(t)$ converge to the semigroup generated by the transition function of the Wiener process in R^3 with generator*

$$G = \frac{c}{3}\Delta \tag{28}$$

where Δ is the three-dimensional Laplace operator.

Remark. Note that for the particular case when the limiting constant $c = 1$, the generator (28) coincides with the evolutionary operator given in Proposition 4.8 of the paper by Pinsky [13] for the dimension $m = 3$.

Proof. The proof of the theorem is similar to that of the planar case. A solution h of the equation

$$\Lambda h = -\mathcal{A}f$$

for any differentiable function (family) f is

$$h = \frac{1}{\lambda}\mathcal{A}f + \frac{1}{4\pi}\int_{S_3} f_\omega\, \mu(d\omega). \tag{29}$$

Since the limiting distribution of the governing Markov process on S_3 is uniform with the density $1/(4\pi)$ then, by formula (12), the projector P is given by

$$Pf = \frac{1}{4\pi}\int_{S_3} f_\omega\, \mu(d\omega). \tag{30}$$

Then, according to (9), (29) and (30), we obtain

$$C_0 f = P\mathcal{A}h = \frac{1}{4\pi}\int_{S_3}\left(\frac{1}{\lambda}\mathcal{A}^2 f + \frac{1}{4\pi}\mathcal{A}\int_{S_3} f_\omega\, \mu(d\omega)\right)\, \mu(d\xi)$$

Using the equality (24) one can easily show that for any $f \in \mathcal{B}$

$$\int_{S_3} \mathcal{A}f\, \mu(d\xi) = \left(\int_0^{2\pi} d\theta \int_0^{\pi} A_{\theta,\nu} \sin\nu\, d\nu\right) f = 0, \tag{31}$$

$$\int_{S_3} \mathcal{A}^2 f\, \mu(d\xi) = \left(\int_0^{2\pi} d\theta \int_0^{\pi} A^2_{\theta,\nu} \sin\nu\, d\nu\right) f = \frac{4\pi v^2}{3}\Delta f \tag{32}$$

where Δ is the three-dimensional Laplacian, and therefore we have

$$C_0 f = \frac{1}{4\pi\lambda}\int_{S_3} \mathcal{A}^2 f\, \mu(d\xi) + \frac{1}{16\pi^2}\int_{S_3} \mathcal{A}\left(\int_{S_3} f_\omega\, \mu(d\omega)\right) \mu(d\xi) = \frac{v^2}{3\lambda}\Delta f.$$

Thus, we obtain

$$C_0 = \frac{v^2}{3\lambda}\Delta$$

and therefore generator (28) under the Kac condition (3) is the limiting operator of the evolution.

Fulfilment of the condition (7) of the Kurtz's theorem is provided by equality (31), and condition (10) can be checked in the same manner as it was done in the planar case.

Therefore, by the Kurtz's approximation theorem, one can conclude that under the Kac condition (3) the distributions of the random evolution $\Xi(t)$ weakly converge in $\mathcal{B}$ to the distribution of the Wiener process in R^3 with generator (28). □

References

[1] Griego R., Hersh R., *Theory of random evolutions with applications to partial differential equations.* – Trans. Amer. Math. Soc., 1971 **156**, p. 405–418.

[2] Hersh R., *Random evolutions: a survey of results and problems.* – Rocky Mount. J. Math., 1974, **4**, 443-496.

[3] Hersh R., Papanicolaou G., *Non-commuting random evolutions and an operator-valued Feynman-Kac formula.* – Comm. Pure Appl. Math., 1972, **25**, p. 337–367.

[4] Hersh R., Pinsky M., *Random evolutions are asymptotically Gaussian.* – Comm. Pure Appl. Math., 1972, **25**, p. 33–44.

[5] Hille E., Phillips R.S., *Functional Analysis and Semigroups.* – Providence, 1957, RI.

[6] Kolesnik A.D., *Weak convergence of a planar random evolution to the Wiener process.* – J. Theoret. Prob., 2001, **14**, p. 485–494.

[7] Kolesnik A.D., Turbin A.F., *The equation of symmetric Markovian random evolution in a plane.* – Stoch. Proc. Appl., 1998, **75**, p. 67–87.

[8] Korolyuk V.S., Swishchuk A.V., *Semi-Markov Random Evolutions.* – Kluwer Publ. House, 1994, Amsterdam.

[9] Kurtz T., *A limit theorem for perturbed operator semigroups with applications to random evolutions.* – J. Func. Anal., 1973, **12**, p. 55–67.

[10] Orsingher E., *Probability law, flow function, maximum distribution of wave-governed random motions and their connections with Kirchoff's laws.* – Stoch. Proc. Appl., 1990, **34**, p. 49–66.

[11] Orsingher E., *Exact joint distribution in a model of planar random motion.* – Stoch. Stoch. Rep., 2000, **69**, p. 1–10.

[12] Pinsky M., *Differential equations with a small parameter and the central limit theorem for functions defined on a finite Markov chain.* – Z. Wahrsch. Verw. Gebiete, 1968, **9**, p. 101–111.

[13] Pinsky M., *Isotropic transport process on a Riemannian manifold.* – Trans. Amer. Math. Soc., 1976 **218**, p. 353–360.

[14] Pinsky M., *Lectures on Random Evolution.* – World Scientific Publ., 1991.

[15] Tolubinsky E.V., *The Theory of Transfer Processes.* – Kiev, Naukova Dumka, 1969 (In Russian).

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