BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(43), 2003, Pages 3–14
ISSN 1024–7696

Bieberbach-Auslander Theorem and Dynamics in Symmetric Spaces

Boris N. Apanasov

Abstract. The aim of this paper (my extended contribution to Intern. Conf. on Discrete Geometry dedicated to A.M.Zamorzaev) is to study dynamics of a discrete isometry group action in a noncompact symmetric space of rank one nearby its parabolic fixed points. Due to Margulis Lemma, such an action on corresponding horospheres is virtually nilpotent, so our extension of the Bieberbach-Auslander theorem for discrete groups acting on connected nilpotent Lie groups can be applied. As result, we show that parabolic fixed points of a discrete group of isometries of such symmetric space cannot be conical limit points and that the fundamental groups of geometrically finite locally symmetric of rank one orbifolds are finitely presented, and the orbifolds themselves are topologically finite.

Mathematics subject classification: 57, 55, 53, 51.
Keywords and phrases: Symmetric spaces, negative curvature, discrete groups, Margulis domain, cusp ends, geometrical finiteness..

1 Introduction

Here we apply our structural theorem on discrete actions on nilpotent groups [6,7] to study the dynamics of a discrete isometry group action nearby its parabolic fixed points at infinity of symmetric spaces of rank 1 with negative curvature. All those spaces X are foliated by horospheres centered at a given point at infinity (i.e. by level surfaces of a Busemann function). The most important case is that of a discrete (parabolic) group $\Gamma \subset \operatorname{Isom} X$ that fixes a point at infinity and preserves setwise each of those horospheres. In this case, by applying the Margulis Lemma, it follows that this discrete parabolic group Γ is virtually nilpotent. Furthermore, at least in symmetric spaces of rank 1 with negative curvature (that is the hyperbolic spaces – either real, complex, quaternionic or octonionic ones), all those horospheres can be identified with a connected simply connected Lie group $\mathcal{N}$ and our discrete group Γ isometrically acts on $\mathcal{N}$ as a subgroup $\Gamma \subset \mathcal{N} \rtimes C$ where C is a compact group of automorphisms of $\mathcal{N}$. In the case of real hyperbolic spaces (of constant negative curvature), horospheres are flat, and discrete Euclidean isometry group actions are described by the Bieberbach theorem [2]. However in the other symmetric spaces of rank 1 this is no longer true. In these spaces horospheres may be represented as non-Abelian nilpotent Lie groups with a left invariant metric, and therefore they

[0]The work was supported in part by the NSF and JSPS through University of Tokyo.

have sectional curvatures of both signs. Here we can use our structural theorem, see [6, 7]:

Theorem 1. *Let $\mathcal{N}$ be a connected, simply connected nilpotent Lie group, C be a compact group of automorphisms of $\mathcal{N}$, and $\Gamma \subset \mathcal{N} \rtimes C$ be a discrete subgroup. Then there exist a connected Lie subgroup $\mathcal{N}_\Gamma$ of $\mathcal{N}$ and a finite index subgroup Γ^* of Γ with the following properties:*

1. *There exists $b \in \mathcal{N}$ such that $b\Gamma b^{-1}$ preserves $\mathcal{N}_\Gamma$;*
2. *$\mathcal{N}_\Gamma / b\Gamma b^{-1}$ is compact;*
3. *$b\Gamma^* b^{-1}$ acts on $\mathcal{N}_\Gamma$ by left translations, and this action is free.*

Here the compactness condition on the group C of automorphisms of $\mathcal{N}$ is essential. The situation when the group C may be noncompact is completely different. For instance, G. Margulis [13] constructed discrete subgroups of $R^3 \rtimes \mathrm{SO}(2,1)$ which are nonabelian free groups, whereas in the compact case any discrete subgroup of $\mathcal{N} \rtimes C$ must be virtually nilpotent, which resembles Gromov's almost flat manifolds [10]. On the other hand, when the group C is compact, there exists a left invariant metric on $\mathcal{N}$ such that $\mathcal{N} \rtimes C$ acts on $\mathcal{N}$ as a group of isometries. So any discrete subgroup of $\mathcal{N} \rtimes C$ can be viewed as a discrete isometry group of $\mathcal{N}$ with respect to some left invariant metric. We remark that our Theorem advances a result by Louis Auslander [1] who proved its claims (1) and (2) only for a finite index subgroup of a given discrete group Γ. In the Euclidean case when $\mathcal{N} = R^n$, this is the Bieberbach theorem, see [2].

A motivation for our study comes from an attempt to understand parabolic (the so-called "thin") ends of negatively curved manifolds, as well as the geometry and topology of geometrically finite pinched negatively curved manifolds, see [2, 3, 5, 9]. The concept of geometrical finiteness first arose in the context of (real) hyperbolic 3-manifolds. Its original definition (due to L.Ahlfors) came from an assumption that such a geometrically finite real hyperbolic manifold M may be decomposed into a cell by cutting along a finite number of its totally geodesic hypersurfaces. Since that time, other definitions of geometrical finiteness have been given by A.Marden, A.Beardon and B.Maskit, and W.Thurston, and the notion has become central to the study of real hyperbolic manifolds. Though other pinched Hadamard manifolds may not have totally geodesic hypersurfaces, the other definitions of geometrical finiteness work in the case of variable negative curvature as well, see [4, 7, 9]. Our previous paper [5] deals with geometrical finiteness in variable curvature in the case of complex hyperbolic manifolds, on the base of a structural theorem for discrete isometric actions on the Heisenberg groups, a predecessor of our Theorem 1. Our proof of Theorem 1 uses different algebraic ideas, see [6, 7].

Here we apply our Theorem 1 in two directions. First we answer a question on dynamics of a discrete isometry group action nearby its limit points, which was left open for variable negative curvature spaces. Namely, it distinguishes two types of limit points of a discrete group $G \subset \operatorname{Isom} X$ acting on a symmetric rank one space X with negative curvature. Namely it shows that parabolic fixed points of such a discrete group G cannot be its conical limit points, i.e. such points $z \in X(\infty)$ that for some (and hence every) geodesic ray ℓ in X ending at z, there is a compact set $K \subset X$ such that the subset $\{g \in G : g(\ell) \cap K \neq \emptyset\}$ is infinite. Such a dichotomy has been recently proved only in the case of real hyperbolic spaces (of constant curvature) by Susskind and Swarup [16] and independently, from a dynamical point of view, by Starkov [15].

The second our result answers another open question (formulated as a conjecture in [9], p.230). Namely, it shows that discrete parabolic groups Γ isometrically acting on a connected Lie groups $\mathcal{N}$ with a compact automorphism group, as well as geometrically finite discrete groups $G \subset \operatorname{Isom} X$ acting on the corresponding symmetric space of rank 1 are finitely presented, and the corresponding quotient orbifolds are topologically finite. Previously, it was known for constant negative curvature. For pinched Hadamard manifolds with various negative curvature, Bowditch [9] proved that such groups are finitely generated. The answer in the case of Heisenberg groups and complex hyperbolic manifolds has been earlier given by the author in [5, 6].

2 Preliminaries

The symmetric spaces of $\mathbb{R}$-rank one of non-compact type are the hyperbolic spaces $H^n_{\mathbb{F}}$, where $\mathbb{F}$ is either the real numbers $\mathbb{R}$, or the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$, or the Cayley numbers $\mathbb{O}$; in last case $n = 2$. They are respectively called as real, complex, quaternionic and octonionic hyperbolic spaces (the latter one $H^2_{\mathbb{O}}$ is also known as the Cayley hyperbolic plane). Algebraically these spaces can be described as the corresponding quotients: $SO(n,1)/SO(n)$, $SU(n,1)/SU(n)$, $Sp(n,1)/Sp(n)$ and $F_4^{-20}/Spin(9)$ where the latter group F_4^{-20} of automorphisms of the Cayley plane $H^2_{\mathbb{O}}$ is the real form of F_4 of rank one. We normalize the metric so the (negative) sectional curvature of $H^n_{\mathbb{F}}$ is bounded from below by -1.

Following Mostow [14] and using the standard involution (conjugation) in $\mathbb{F}$, $z \to \bar{z}$, one can define projective models of the hyperbolic spaces $H^n_{\mathbb{F}}$ as the set of negative lines in the Hermitian vector space $\mathbb{F}^{n,1}$, with Hermitian structure given by the indefinite $(n,1)$-form

$$\langle\langle z, w \rangle\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n - z_{n+1}\overline{w}_{n+1}\,.$$

Here, taking non-homogeneous coordinates, one can obtain unit ball models (in the unit ball $B^n_{\mathbb{F}}(0,1) \subset \mathbb{F}^n$) for the first three spaces. Since the multiplication by quaternions is not commutative, we specify that we use “left” vector space $\mathbb{H}^{n,1}$ where the multiplication by quaternion numbers is on the left. However, it does not work for the Cayley plane since $\mathbb{O}$ is non-associative, and one should use a Jordan algebra of 3×3 Hermitian matrices with entries from $\mathbb{O}$ whose group of automorphisms is F_4, see [14].

Another models of $H^n_{\mathbb{F}}$ use the so called horospherical coordinates [6, 11] based on foliations of $H^n_{\mathbb{F}}$ by horospheres centered at a fixed point ∞ at infinity $\partial H^n_{\mathbb{F}}$

which is homeomorphic to $(n \dim_{\mathbb{R}} \mathbb{F} - 1)$-dimensional sphere. Such a horosphere can be identified with the nilpotent group N in the Iwasawa decomposition KAN of the automorphism group of $H^n_{\mathbb{F}}$. The nilpotent group N can be identified with the product $\mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$ (see [14]) equipped with the operations:

$$(\xi, v) \cdot (\xi', v') = (\xi + \xi', v + v' + 2 \operatorname{Im}\langle \xi, \xi' \rangle) \quad \text{and} \quad (\xi, v)^{-1} = (-\xi, -v)\,,$$

where $\langle , \rangle$ is the standard Hermitian product in $\mathbb{F}^{n-1}$, $\langle z, w \rangle = \sum z_i \overline{w_i}$. The group N is a 2-step nilpotent Carnot group with center $\{0\} \times \operatorname{Im} \mathbb{F} \subset \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$, and acts on itself by the left translations $T_h(g) = h \cdot g\,, \quad h, g \in N$.

Now we may identify

$$H^n_{\mathbb{F}} \cup \partial H^n_{\mathbb{F}} \backslash \{\infty\} \longrightarrow N \times [0, \infty) = \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F} \times [0, \infty)\,,$$

and call this identification the *"upper half-space model"* for $H^n_{\mathbb{F}}$ with the natural horospherical coordinates (ξ, v, u). In these coordinates, the above left action of N on itself extends to an isometric action (Carnot translations) on the $\mathbb{F}$-hyperbolic space in the following form:

$$T_{(\xi_0, v_0)} : (\xi, v, u) \longmapsto (\xi_0 + \xi\,, v_0 + v + 2 \operatorname{Im}\langle \xi_0, \xi \rangle\,, u)\,,$$

where $(\xi, v, u) \in \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F} \times [0, \infty)$.

There are a natural norm and an induced by this norm distance on the Carnot group $N = \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$, which are known in the case of the Heisenberg group (when $\mathbb{F} = \mathbb{C}$) as the Cygan's norm and distance. Using horospherical coordinates, they can be extended to a norm on $H^n_{\mathbb{F}}$, see [6]:

$$|(\xi, v, u)|_c = |\,(|\xi|^2 + u - v)|^{1/2}\,, \tag{1}$$

where $|.|$ is the norm in $\mathbb{F}$, and to a metric ρ_c (still called the Cygan metric) on $\mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F} \times [0, \infty) = \overline{X} \backslash \{\infty\}$:

$$\rho_c\big((\xi, v, u), (\xi', v', u')\big) = \big|\, |\xi - \xi'|^2 + |u - u'| - (v - v' + 2 \operatorname{Im}\langle \xi, \xi' \rangle)\big|^{\frac{1}{2}}\,. \tag{2}$$

It follows directly from the definition that Carnot translations and rotations are isometries with respect to the Cygan metric ρ_c. Moreover, the restrictions of this metric to different horospheres centered at ∞ are the same, so Cygan metric plays the same role as Euclidean metric does on the upper half-space model for the real hyperbolic space $\mathbb{H}^n$.

The group of automorphisms of $H^n_{\mathbb{H}}$ is $PSp(n, 1)$. The stabilizer K of the origin is $Sp(1) \times Sp(n)$ which can be described in the matrix form as:

$$\begin{bmatrix} M & 0 \\ 0 & \nu \end{bmatrix}$$

where $M \in Sp(n)$ and $\nu \in Sp(1)$. Note the matrix acts on the right, and the projectivization is given by multiplication on the left. So in the ball model, the action is:

$$z \to \nu^{-1} z M\,.$$

The stabilizer of a real geodesic connecting two points $(0,1)$ and $(0,-1)$ is $MA = Sp(1) \times Sp(n-1) \times \mathbb{R}$. This action can be described in the matrix form as:

$$\begin{bmatrix} M & 0 & 0 \\ 0 & \nu\cosh r & \nu\sinh r \\ 0 & \nu\sinh r & \nu\cosh r \end{bmatrix}$$

where $M \in Sp(n-1)$, $\nu \in Sp(1)$ and $r \in \mathbb{R}$. Specially $Sp(1)$ acts as

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

If $(0, \mathbb{H})$ is the $\mathbb{H}$-line containing the real geodesic joining $(0,1)$ and $(0,-1)$, the action of $Sp(1)$ on this $\mathbb{H}$-line is:

$$(0, \mathbb{H}) \to \nu^{-1}(0, \mathbb{H}\nu, \nu) = (0, \nu^{-1}\mathbb{H}\nu).$$

But in general, $\nu \in Sp(1)$ maps (z, z_n) to $(\nu^{-1}z\,, \nu^{-1}z_n\nu)$.

A Cayley number $z \in \mathbb{O}$ is a pair of quaternions, $z = (q_1, q_2)$, and the multiplication in $\mathbb{O}$ is given by

$$(q_1, q_2)(p_1, p_2) = (q_1p_1 - \bar{p_2}q_2, p_2q_1 + q_2\bar{p_1})\,.$$

The standard involution (conjugation) in $\mathbb{O}$ is defined by $\overline{(q_1, q_2)} = (\bar{q_1}, -q_2)$, so for $z = (q_1, q_2) \in \mathbb{H} \times \mathbb{H} = \mathbb{O}$, we have $\operatorname{Im} z = (\operatorname{Im} q_1, q_2)$ and $\operatorname{Re} z = \operatorname{Re} q_1$. Then Cayley numbers satisfy the usual properties like: $x\bar{x} = |x|^2$, $|xy| = |x||y|$, $x^{-1} = \bar{x}/|x|^2$, $\overline{xy} = \bar{y}\bar{x}$. Even though Cayley numbers are not commutative, nor associative, by Artin's lemma a subalgebra generated by two elements is associative. Cayley hyperbolic plane is made out of an exceptional Jordan algebra of 3×3 Hermitian matrices with entries from $\mathbb{O}$ whose group of automorphisms is F_4, see [14]. The group of automorphisms of the Cayley plane $H^2_{\mathbb{O}}$ is F_4^{-20}, the real form of F_4 of rank one. The stabilizer in F_4^{-20} of the origin $(0,0) \in B^2_{\mathbb{O}}(0,1) = H^2_{\mathbb{O}}$ is $Spin(9)$ operating on $\mathbb{O}^2 = \mathbb{R}^{16}$ via the spinor representation. If $L_1 = \mathbb{O} \times 0$ and $L_2 = 0 \times \mathbb{O}$ denote the coordinate $\mathbb{O}$-axes, then the stabilizer of L_1 acts on L_1 as $SO(8)$ via the even $\frac{1}{2}$-spin representation, and on L_2 as odd $\frac{1}{2}$-spin representation. The stabilizer of the real line through $(0,0)$ and $(1,0)$ is Spin(7).

3 Margulis region and parabolic cusps

One of the most important tools for studying negatively curved spaces is given by the Margulis Lemma which induces the thick-thin decomposition of corresponding orbifolds, see [8, 9]. Such orbifolds are quotients $M = X/G$ of symmetric spaces X by discrete isometric actions of their fundamental groups $\pi_1^{orb} \cong G \subset \operatorname{Isom} X$. Adding the induced discrete action of G in some domain at infinity ∂X, we obtain a partial closure $M(G)$ of that orbifold M. More precisely, let $\Lambda(G) \subset \partial X$ and $\omega(G) = \partial X \backslash \Lambda(G)$ be the *limit and discontinuity sets* of $G \subset \operatorname{Isom} X$. Then we set $M(G) = (X \cup \Omega(G))/G$.

Let ϵ be a positive number less than $\epsilon(n)$, the Margulis constant for symmetric n-spaces of rank one. For a given discrete group $G \subset \operatorname{Isom} X$ and its orbifold $M = X/G$, we define the ϵ-thin part $\operatorname{thin}_\epsilon(M)$ as

$$\operatorname{thin}_\epsilon(M) = \{x \in X : G_\epsilon(x) = \langle g \in G : d(x, g(x)) < \epsilon\rangle \text{ is infinite}\}/G\,.$$

The thick part $\operatorname{thick}_\epsilon(M)$ of M is defined as the closure of the complement to the thin part, $\operatorname{thin}_\epsilon(M) \subset M$.

As a consequence of the Margulis Lemma, there is the following description of the thin part of M [8, 9]:

Theorem 2. *Let $G \subset \operatorname{Isom} X$ be a discrete group and ϵ, $0 < \epsilon < \epsilon(n)$, be chosen. Then the ϵ-thin part $\operatorname{thin}_\epsilon(M)$ of $M = X/G$ is a disjoint union of its connected components, and each such component has the form $T_\epsilon(\Gamma)/\Gamma$ where Γ is a maximal infinite elementary subgroup of G. Here, for each such elementary subgroup $\Gamma \subset G$, the connected component (Margulis region)*

$$T_\epsilon = T_\epsilon(\Gamma) = \{x \in X : \Gamma_\epsilon(x) = \langle g \in \Gamma : d(x, \gamma(x)) < \epsilon\rangle \textit{ is infinite}\}$$

is precisely invariant with respect to the subgroup Γ in G:

$$\Gamma(T_\epsilon) = T_\epsilon\,, \quad g(T_\epsilon) \cap T_\epsilon = \emptyset \quad \textit{for any } g \in G \backslash \Gamma\,.$$

We note that in the real hyperbolic case of dimension 2 and 3, a Margulis region T_ϵ with parabolic stabilizer $\Gamma \subset G$ can be taken as a horoball neighborhood centered at the parabolic fixed point p, $\Gamma(p) = p$. It is not true in general due to Apanasov's construction in real hyperbolic spaces of dimension at least 4, see [2]. As we discussed it in [5], this construction works in complex hyperbolic spaces $\operatorname{ch} n$ as well as in other rank one symmetric spaces X. However, we may apply our Theorem 1 to describe parabolic Margulis regions in all such spaces.

Namely, let $\Gamma \subset G$ be a discrete parabolic subgroup. We may view X from the fixed point $p \in \partial X$ in the way we have used in §2 to define the upper half-space model for X. Then, by using the foliation of X by horospheres X_t centered at p, we identify $\overline{X} \backslash \{p\}$ and $\mathcal{N} \times [0, \infty)$, where $X_t \cong \mathcal{N}$ is a connected, simply connected Lie group with a compact automorphism group C. Since the parabolic group Γ acts on each horosphere X_t centered at the fixed point p as a discrete subgroup of $\mathcal{N} \rtimes C$, we can apply Theorem 1 which implies that there exists a Γ-invariant connected subspace $\sigma \subset \partial X \backslash \{p\} \cong \mathcal{N}$ where Γ acts co-compactly. Also we have a finite index subgroup $\Gamma^* \subset \Gamma$ which acts on σ freely by left translations. In fact, σ is a translate of a connected Lie subgroup $\mathcal{N}_\Gamma$ of $\partial X \backslash \{p\} \cong \mathcal{N}$. Now we define the subspace $\tau \subset X$

to be spanned by σ and all geodesics $(z,p) \subset X$ connecting $z \in \sigma$ to the parabolic fixed point p. Let τ_t be the "half-plane" in τ of a height $t > 0$, that is the part of τ whose last horospherical coordinate is at least t.

Lemma 3. *Let $G \subset \operatorname{Isom} X$ be a discrete group in a rank one symmetric space X and $p \in \partial X$ a parabolic fixed point of G. Let T_ϵ be a Margulis region for p and let τ_t be the half-plane defined as above. Then for any δ, $0 < \delta < \epsilon/2$, there exists a positive number $t > 0$ such that the Margulis region T_ϵ contains the δ-neighborhood $N_\delta(\tau_t)$ of the half-plane τ_t.*

Proof: Let $\Gamma \subset G$ be the maximal parabolic subgroup fixing a given parabolic fixed point $p \in \partial X$. Since Γ preserves the subspace $\sigma \subset \partial X \backslash \{p\}$, it preserves the boundary $\partial\tau_t$ of each half-plane τ_t.

As it was shown in [12], the geometry of horospheres in the space X with sectional curvatures $-1 \leq K \leq -1/4$ may be closely compared with that in the spaces of constant negatives curvature $-1/4$ and -1, respectively. In particular, for two asymptotic geodesic rays ℓ and ℓ' approaching $p \in \partial X$ from two points x and x' on the same horosphere, with a horospherical distance R_0 between them, we have:

$$(2\operatorname{arcsinh}(2R_0))\, e^{-t} \leq d(\ell(t), \ell'(t)) \leq R_0 e^{-\frac{t}{2}}\,.$$

This implies that distances on horospheres in X of height t exponentially decrease as t goes to $+\infty$. On the other hand, due to Theorem 1, infinite order elements $\gamma \in \Gamma$ act on the boundary $\partial\tau_t$, $t > 0$, as virtual translations, and the quotient $\partial\tau_t/\Gamma$ is compact. Therefore, for positive numbers δ and ϵ', $2\delta + \epsilon' < \epsilon$, there exist some height $t_{\epsilon'}$ such that

$$\partial\tau_t \subset T_{\epsilon'}(\Gamma) \subset T_{\epsilon'}(G) = T_{\epsilon'} \quad \text{for all } \ t > t_{\epsilon'}\,.$$

Clearly, the same is true for the whole half-plane:

$$\tau_t \subset T_{\epsilon'}(\Gamma) \subset T_{\epsilon'}. \tag{3}$$

Now, for any $x \in N_\delta(\tau_t)$ with $t > t_{\epsilon'}$, we have a δ-close point $x_0 \in \tau_t$, $d(x, x_0) < \delta$. Due to 3, there is an infinite order element $\gamma \in \Gamma$ such that $d(x_0, \gamma(x_0)) < \epsilon'$. It implies:

$$d(x, \gamma(x)) \leq d(x, x_0) + d(x_0, \gamma(x_0)) + d(\gamma(x_0), \gamma(x) < 2\delta + \epsilon' < \epsilon\,,$$

which shows that the point x and thus the whole δ-neighborhood $N_\delta(\tau_t)$ belong to the Margulis region T_ϵ. ■

Now we can (negatively) answer the question of *whether a parabolic fixed point of a discrete group $G \subset \operatorname{Isom} X$ may also be its conical limit point.*

Here a limit point $z \in \Lambda(G)$ is called a *conical limit point* of a discrete group $G \subset \operatorname{Isom} X$ if, for some (and hence every) geodesic ray $\ell \subset X$ ending at z, there is a compact set $K \subset X$ such that $g(\ell) \cap K \neq \emptyset$ for infinitely many elements $g \in G$.

This definition is equivalent to a possibility to approximate the limit point $z \in \Lambda(G)$ by a G-orbit $\{g_i(x)\}$ of a point $x \in X$ inside a tube (cone) in X with vertex $z \in \partial X$. Applying an argument originally due to A.Beardon and B.Maskit, one can use the following equivalent definition of conical limit points [7]:

Lemma 4. *A point $z \in \Lambda(G)$ is a conical limit point of a discrete group $G \subset \operatorname{Isom} X$ in a negatively curved space X if and only if, for every geodesic ray $\ell \subset X$ ending at z and for every $\delta > 0$, there is a point $x \in X$ and a sequence of distinct elements $g_i \in G$ such that the orbit $\{g_i(x)\}$ approximates z inside the δ-neighborhood $N_\delta(\ell)$ of the ray ℓ.*

There are other (equivalent) definitions of conical limit points [9]. One of them is even intrinsic to the action of the group G on the limit set $\Lambda(G)$. Namely, $z \in \Lambda(G)$ is a conical limit point if there is a sequence $\{g_i\}$ of distinct elements of G such that, for any other limit point $y \in \Lambda(G)\backslash\{z\}$, the sequence of pairs $(g_i^{-1}(z), g_i^{-1}(y))$ lies in a compact subset of $(\Lambda(G) \times \Lambda(G))\backslash\Delta(\Lambda)$, where $\Delta(\Lambda) = \{(x,x) : x \in \Lambda(G)\}$.

Theorem 5. *Let $G \subset \operatorname{Isom} X$ be a discrete group in a rank one symmetric space X. Then any parabolic fixed point of G cannot be its conical limit point.*

Proof: Let $\Gamma \subset G$ be the maximal parabolic subgroup of given group G fixing a parabolic fixed point $p \in \partial X$. As in Lemma 3, viewing X from the point p at infinity by using horospherical coordinates and applying Theorem 1, we again have a Γ-invariant connected subspace $\sigma \subset \partial X\backslash\{p\}$ where Γ acts co-compactly, and on which a finite index subgroup $\Gamma^* \subset \Gamma$ acts freely by left translations. Applying Lemma 3 to the subspace $\tau \subset X$ spanned by σ and p, we have positive numbers δ and t so that the δ-neighborhood $N_\delta(\tau_t)$ of the half-plane τ_t is contained in the parabolic Margulis region T_ϵ at p.

Now suppose that the point p is also a conical limit point of G. Then for a geodesic ray $\ell \subset \tau_t$ tending to p, there must exist a point $x \in X$ and a sequence of distinct elements $g_i \in G$ such that the sequence $g_i(x)$ tends to p inside of δ-neighborhood $N_\delta(\ell)$ of the ray ℓ, see Lemma 4. However, due to Lemma 3, $N_\delta(\ell) \subset N_\delta(\tau_t) \subset T_\epsilon$. Since the Margulis region T_ϵ is precisely invariant for the subgroup $\Gamma \subset G$ (Theorem 2), it follows that all elements g_i belong in fact to the parabolic subgroup Γ. Hence all g_i preserve each horosphere X_t centered at p. Using compactness of $\partial\tau_t/\Gamma$, we see then that all points $g_i(x)$ must lie in a compact part of $N_\delta(\tau_t)$ and hence cannot approach the limit point p. This contradiction completes the proof. ■

Now we shall apply our structural Theorem 1 to clarify the structure of cusp ends of geometrically finite locally symmetric rank one manifolds/orbifolds. This new geometric insight on dynamics of discrete isometry group actions near their parabolic fixed points will allow us to prove that fundamental groups of such manifolds/orbifolds are in fact finitely presented.

A parabolic fixed point $p \in \partial X$ of a discrete group $G \subset \operatorname{Isom} X$ in a pinched negatively curved space X is called a *cusp point* if the quotient $(\Lambda(G) \setminus \{p\})/G_p$ of the limit set of G by the action of the parabolic stabilizer $G_p = \{g \in G : g(p) = p\}$ is compact [9].

This leads to a definition (**GF1**, originally due to A.Beardon and B.Maskit, see [2]) of *geometrically finite* discrete groups $G \subset \operatorname{Isom} X$ (and their negatively curved orbifolds $M = X/G$) as those whose limit set $\Lambda(G) \subset \partial X$ entirely consists of conical limit points and parabolic cusps.

Another definition of geometrical finiteness (**GF2**, originally due to Albert Marden, see [2]) is that the quotient $M(G) = X \cup \Omega(G)/G$ has only finitely many topological ends and each of these ends can be identified with the end of $M(\Gamma)$, where Γ is a maximal parabolic subgroup of G.

Additional two definitions of geometrical finiteness are originally due to W.Thurston, see [2]:

(GF3): The thick part of the minimal convex retract (=convex core) $C(G)$ of X/G is compact.

(GF4): For some $\epsilon > 0$, the uniform ϵ-neighborhood of the convex core $C(G) \subset X/G$ has finite volume, and there is a universal bound on the orders of finite subgroups in G.

Theorem 6. [9] *Let X be a pinched Hadamard manifold. Then the four definitions* **GF1, GF2, GF3** *and* **GF4** *of geometrical finiteness for a discrete group $G \subset \operatorname{Isom} X$ are all equivalent.*

We shall add that in contrast to the real hyperbolic geometry, our examples [5] of discrete parabolic groups acting in complex hyperbolic space suggest that there exists no elegant formulation of geometrical finiteness involving finite-sided polyhedra.

Now we shall give a new geometric definition of parabolic cusp points (cusp ends) for discrete isometry groups acting in non compact symmetric spaces X of rank one. Let a point $p \in \partial X$ be a parabolic fixed point of a discrete group $G \subset \operatorname{Isom} X$ and let $\Gamma = G_p$ be the stabilizer of p in G, that is a maximal parabolic subgroup in G with fixed point p. As before, taking horospherical coordinates on X with respect to $p \in \partial X$, we can regard this stabilizer as $\Gamma \subset \mathcal{N} \rtimes C$ where C is a compact automorphism group of the connected Lie group $\mathcal{N}$ representing horospheres in X. Let ρ_c be the ($\mathcal{N} \rtimes C$-invariant) Cygan metric on $\mathcal{N} \times [0, \infty) = X \cup \partial X \backslash \{p\}$ defined in (2), and let $\mathcal{N}_\Gamma \subseteq \mathcal{N} = \partial X \backslash \{p\}$ be a minimal connected subgroup of the nilpotent group $\mathcal{N}$ given by Theorem 1. The parabolic stabilizer Γ preserves $\mathcal{N}_\Gamma$ and acts there cocompactly.

Definition 7. *Given a positive number δ and a parabolic fixed point $p \in \partial X$ of a discrete group $G \subset \operatorname{Isom} X$ with stabilizer $\Gamma = G_p \subset G$, the set*

$$U_{p,\delta} = \{x \in X \cup \partial X \backslash \{p\} : \rho_c(x, \mathcal{N}_\Gamma) \geq \frac{1}{\delta}\} \tag{4}$$

is called a (closed) standard cusp neighborhood of radius $\delta > 0$ at p, provided it is precisely invariant with respect to the stabilizer Γ in G:

$$\gamma(U_{p,\delta}) = U_{p,\delta} \qquad \textit{for} \quad \gamma \in \Gamma = G_p\,,$$

$$g(U_{p,\delta}) \cap U_{p,\delta} = \emptyset \qquad \textit{for} \quad g \in G\backslash G_p\,.$$

Lemma 8. *Let $p \in \partial X$ be a parabolic fixed point of a discrete group $G \subset \operatorname{Isom} X$ in a rank one symmetric space X. Then p is a parabolic cusp point if and only if it has a standard cusp neighborhood $U_{p,\delta}$.*

Proof: As before, let $\Gamma \subset G$ be the parabolic stabilizer of a given parabolic fixed point p, and $\mathcal{N}_\Gamma \subseteq \mathcal{N} = \partial X\backslash\{p\}$ be the minimal connected Γ-invariant subspace of the nilpotent group $\mathcal{N}$ given by Theorem 1. If p has a standard cusp neighborhood $U_{p,\delta} \subset \overline{X}\backslash\{p\}$ then the limit set $\Lambda(G)$ must lie in its complement $\partial X\backslash U_{p,\delta}$ due to the condition of its precise Γ-invariantness. Hence $\Lambda(G)\backslash\{p\}/\Gamma$ is compact because of compactness of $\mathcal{N}_\Gamma/\Gamma$ (due to Theorem 1). The converse statement follows from Bowditch's arguments in the proof [9] of Theorem 6. ■

For a given discrete group $\Gamma \subset \mathcal{N} \rtimes C \subset \operatorname{Isom} X$, the quotient space $M(\Gamma) = (X \cup \partial X\backslash\{\infty\})/\Gamma$ has a unique end. We call this end a *standard parabolic end* with $(X, \operatorname{Isom} X)$-geometry. It is clear that (closed) neighborhoods of a standard parabolic end may be taken as $U_{\infty,\delta}/\Gamma$, $\delta > 0$.

Applying the above definitions of cusp points and ends, Lemma 8 and Theorem 6, we see that for a cusp point $p \in \partial X$ of a geometrically finite discrete group $G \subset \operatorname{Isom} X$, the family $E_p = \{U_{p,\delta}/G_p\}$ of closed subspaces in $M(G)$ naturally defines the cusp end of $M(G)$ identified by the G-orbit of the parabolic cusp point p. It is isometric to a standard cusp end, actually to the end of $M(G_p)$.

We may represent a standard cusp neighborhood U_{p,δ_0} at a cusp point p of a discrete group $G \subset \operatorname{Isom} X$ as the product

$$U_{p,\delta_0} = S_{p,\delta_0} \times (0, r_0]\,, \tag{5}$$

if we foliate U_{p,δ_0} by subsets $S_{p,\delta}\,, 0 < \delta \le \delta_0\,,$ of the form:

$$S_{p,\delta} = \{x \in X \cup \partial X\backslash\{p\} : \rho_c(x, \mathcal{N}_{G_p}) = 1/\delta\}\,. \tag{6}$$

Since each set $S_{p,\delta}$ is G_p-invariant, we see that the standard cusp neighborhood $U_{p,\delta_0}/G_p \subset M(G)$ of the cusp end E_p in the orbifold $M(G)$ is the product $(S_{p,\delta_0}/G_p) \times (0,1]$. Furthermore, due to compactness of the automorphism group C of the nilpotent group $\mathcal{N}$, this foliation of a standard cusp neighborhood U_{p,δ_0} by G_p-invariant sets $S_{p,\delta}$ defines a G_p-equivariant retraction

$$R_p : \quad U_{p,\delta_0} \longrightarrow \mathcal{N}_{G_p}\,. \tag{7}$$

This retraction shows topological finiteness of ends of noncompact orbifolds $\mathcal{N}/\Gamma$ for discrete parabolic groups $\Gamma \subset \mathcal{N} \rtimes C$ (and, with a little bit more work, existence of a vector bundle structure on them, compare our Theorem 4.1 in [5]), as well as topological finiteness of cusp ends of $(X, \operatorname{Isom} X)$-orbifolds. So, due to Theorem 1, all those ends have the homotopy type of closed virtually nilpotent orbifolds $\mathcal{N}_\Gamma/\Gamma$. This, together with Theorem 6, completes the proof of the following fact:

Theorem 9. *For any geometrically finite discrete group $G \subset \operatorname{Isom} X$ in a symmetric rank one non-compact space X, the orbifold $M = X/G$ is topologically finite. In other words, M is orbifold-homeomorphic to the interior of a compact orbifold with boundary obtained from $M(G)$ by gluing to its ends closed virtually nilpotent orbifolds of the form $\mathcal{N}_\Gamma/\Gamma$ where $\Gamma \subset \mathcal{N} \rtimes C$ is a parabolic discrete group in the corresponding nilpotent group $\mathcal{N}$ representing horospheres in X.*

It immediately implies:

Corollary 10. *Let $\mathcal{N}$ be a nilpotent group representing horospheres in a symmetric rank one non-compact space X and C its compact group of automorphisms. Then all discrete parabolic groups $\Gamma \subset \mathcal{N} \rtimes C$ as well as geometrically finite groups $G \subset \operatorname{Isom} X$ are finitely presented.*

References

[1] Auslander L., *Bieberbach's theorem on space groups and discrete uniform subgroups of Lie groups, II.* – Amer. J. Math., 1961, **83**, p. 276–280.

[2] Apanasov B., *Conformal geometry of discrete groups and manifolds.* – De Gruyter Expositions in Math., W. de Gruyter, Berlin- New York, 2000, XIV + 523 pp.

[3] Apanasov B., *Geometrically finite hyperbolic structures on manifolds.* – Ann. of Glob. Analysis and Geom., 1983, **1:3**, p. 1–22.

[4] Apanasov B., *Geometry and topology of complex hyperbolic and CR-manifolds.* – Russian Math. Surveys, 1997, **52:5**, p. 895–928.

[5] Apanasov B., Xiangdong Xie, *Geometrically finite complex hyperbolic manifolds.* – Intern. J. of Math., 1997, **8**, p. 703–757.

[6] Apanasov B., Xiangdong Xie, *Nilpotent groups and manifolds of negative curvature.* – Topics in Complex Analysis, Differential Geometry and Mathematical Physics (Proc. of the third Intern. Workshop at Varna, K.Sekigawa and S.Dimiev, Eds), World Scientific, Singapore, 1997, p. 105–115.

[7] Apanasov B., Xiangdong Xie, *Discrete actions on nilpotent groups and negatively curved spaces.* – Diff. Geom. Appl., to appear.

[8] Ballmann W., Gromov M., Schroeder V., *Manifolds of nonpositive curvature.* – Birkhäuser, Boston, 1985.

[9] Brian Bowditch, *Geometrical finiteness with variable negative curvature.* – Duke J. Math., 1995, **77**, p. 229–274.

[10] Buser P., Karcher H., *Gromov's almost flat manifolds.* – Asterisque, 1981, **81**, p. 1–148.

[11] Goldman W., Parker J., *Dirichlet polyhedron for dihedral groups acting on complex hyperbolic space.* – J. of Geom. Analysis, 1992, **2:6**, p. 517–554.

[12] Heintze Ernst, Hof Hans-Christoph, *Geometry of horospheres.* – J.Diff. Geom., 1977, **12**, p. 481–491.

[13] Margulis G.A., *Free properly discontinuous groups of affine transformations.* – Dokl. Akad. Nauk SSSR, 1983, **272**, p. 937–940 (In Russian).

[14] Mostow G.D., *Strong rigidity of locally symmetric spaces.* – Princeton Univ. Press, 1973.

[15] Starkov A.N., *Parabolic fixed points of Kleinian groups and the horospherical foliation on hyperbolic manifolds.* – Intern. J. Math., 1997, **8**, p. 289–299.

[16] Susskind Perry, Swarup Gadde A., *Limit sets of geometrically finite Kleinian groups.* – Amer. J. Math., 1992, **114**, p. 233–250.

Boris N. Apanasov
Department of Mathematics
University of Oklahoma
Norman, OK 73019-0351, USA
E-mail: *apanasov@ou.edu*

Received October 25, 2002