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The commutative Moufang loops with minimum conditions for subloops I

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Abstract. The structure of the commutative Moufang loops (CML) with minimum condition for subloops is examined. In particular it is proved that such a CML Q is a finite extension of a direct product of a finite number of the quasicyclic groups, lying in the centre of the CML Q. It is shown that the minimum conditions for subloops and for normal subloops are equivalent in a CML. Moreover, such CML also characterized by different conditions of finiteness of its multiplicative groups.

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The loop Q satisfies *minimum condition for subloops with the property* α if any decreasing chain of its subloops with the property α $H_1 \supseteq H_1 \supseteq \ldots$, i.e. $H_n = H_{n+1} = \ldots$ for a certain n. In this paper the construction of the commutative Moufang loops (abbreviated CMLs) with minimum condition for subloops is examined. In particular, it is shown that such a CML Q decomposes into a direct product of finite number of quasicyclic groups which lies in the centre of Q, and a finite CML (Section 2). In the third Section these loops are described with the help of their multiplicative groups. Finally, it is shown in the fourth section that for the CML, the minimum conditions for subloops are equivalent to the minimum condition for normal subloops, and in the case of ZA-loops these conditions are equivalent to the minimum condition for normal associative subloops. It follows from the last statement that the infinite commutative Moufang ZA-loop Q has an infinite centre and if the centre of the CML satisfies the minimum condition for the subloops, then Q itself satisfies this condition.

We finally note that loops, in particular the CML, with different conditions of finiteness are examined in [1–3]. We remind that the condition of finiteness means such's property, that holds true for all finite loops, but there exist infinite loops that do not have this property.

1 Preliminaries

Let us bring some notions and results on the theory of the commutative Moufang loops from [4]. A *commutative Moufang loop* (abbreviated CML) is characterized by the identity

$$x^2 \cdot yz = xy \cdot xz. \tag{1.1}$$

The *multiplicative group* $\mathfrak{M}(Q)$ of the CML Q is the group generated by all the *translations* $L(x)$, where $L(x)y = xy$. The subgroup $I(Q)$ of the group $\mathfrak{M}(Q)$ generated by all the *inner mappings* $L(x,y) = L^{-1}(xy)L(x)L(y)$ is called the *inner mapping group* of the CML Q. The subloop H of the CML Q is called *normal* (*invariant*) in Q if $I(Q)H = H$.

Lemma 1.1 [4]. *Let Q be a commutative Moufang loop with the multiplicative group $\mathfrak{M}$. Then $\mathfrak{M}/Z(\mathfrak{M})$, where $Z(\mathfrak{M})$ is the centre of the group $\mathfrak{M}$, and $\mathfrak{M}' = (\mathfrak{M}, \mathfrak{M})$ are locally finite 3-groups and will be finite if Q is finitely generated.*

The *associator* (a,b,c) of the elements a,b,c of the CML Q are defined by the equality $ab \cdot c = (a \cdot bc)(a,b,c)$. The identities

$$L(x,y)z = z(z,y,x), \tag{1.2}$$

$$(x^p, y^r, z^s) = (x,y,z)^{prs}, \tag{1.3}$$

$$(x,y,z)^3 = 1, \tag{1.4}$$

$$(xy,u,v) = (x,u,v)((x,u,v),x,y)(y,u,v)((y,u,v),y,x) \tag{1.5}$$

hold in the CML [4].

The *centre* $Z(Q)$ of the CML Q is a normal subloop $Z(Q) = \{x \in Q | (x,y,z) = 1 \forall y,z \in Q\}$.

Lemma 1.2 [4]. *In a commutative Moufang loop Q the following statements hold true:*

1) for any $x \in Q$ $x^3 \in Z(Q)$;

2) the quotient loop $Q/Z(Q)$ has the index three.

Lemma 1.3 [4]. *The periodic commutative Moufang loop is locally finite.*

Lemma 1.4 [5]. *The periodic commutative Moufang loop Q decomposes into a direct product of its maximum p-subloops Q_p, in addition Q_p belongs to the centre $Z(Q)$ under $p \neq 3$.*

The *system* σ of the normal subloops of the loop Q is called *normal* if it:

1) contains the loop Q and its identity subloop;

2) is linearly ordered by the inclusion;

3) the intersection and union of any non-empty set of elements of σ is an element of σ (fullness).

If $A \subseteq B$ are two members of the system σ and between them there are no other members of this system then it is said that the subloops A and B form a *jump* in the system σ. The quotient loop B/A is called the *factor* of this system. The normal system σ is called *central* if for any jump A and B of the system σ, $B/A \subseteq Z(B/A)$.

The loop possessing a central system is called a *Z-loop*. This statement is proved in [4, Theorems 4.1, Chap. VI; 10.1, Chap. VIII].

Lemma 1.5. *Any commutative Moufang loop is a Z-loop.*

If the loop possesses a central system entirely ordered by the inclusion (the *central series*), then this loop is called *ZA-loop*.

Lemma 1.6 [3]. *Any normal different from the identity element subloop H of the commutative Moufang ZA-loop Q has a different from identity element intersection with its centre.*

If the upper central series of the ZA-loop have a finite length, then the loop is called *centrally nilpotent*. The least of such length is called the *class* of the central nilpotentcy.

Lemma 1.7 [3]. *If a commutative Moufang ZA-loop Q has an infinite associative normal subloop, then its centre $Z(Q)$ is infinite.*

Lemma 1.8 (Bruck-Slaby Theorem) [4]. *The finitely generated commutative Moufang loop is centrally nilpotent.*

Lemma 1.9 [3]. *If at least one maximal associative subloop of the commutative Moufang loop Q satisfies the minimum conditions for subloops, then Q satisfies these conditions itself.*

The CML Q will be called *divisible* it the equality $x^n = a$ has at least one solution in Q, for any number $n > 0$ and any element $a \in Q$. If $n = 3$, then $a = b^3 \in Z(Q)$ by Lemma 1.2. Therefore it takes place.

Lemma 1.10. *If a subloop of the commutative Moufang loop Q is divisible, it belongs to the centre $Z(Q)$ and, consequently, is normal in Q.*

The *quasicyclic p-groups* are some important examples of divisible CML. As abstract groups they have the set of generators $1 = a_0, a_1, a_2, \ldots, a_n, \ldots$ and defining relations $a_0 = a_1^p, a_1 = a_2^p, \ldots, a_n = a_{n+1}^p, \ldots$.

A CML is called *injective* if there exists a homomorphism $\gamma : B \to Q$ such that $\alpha\gamma = \beta$, for any monomorphism $\alpha : A \to B$ and homomorphism $\beta : A \to Q$.

Lemma 1.11. *The divisible commutative Moufang loops are injective.*

Proof. By Lemma 1.10 a divisible CML is associative, but divisible abelian groups are injective [6].

Further we will denote by $< M >$ the subloop of loop Q, generated by the set $M \subseteq Q$.

Proposition 1.12. *The divisible subloop D of the commutative Moufang loop Q serves as a direct factor for Q, i.e. $Q = D \times C$ for a certain subloop C of the loop Q. We can choose such a subloop that it possesses the given before subloop B of the loop Q for which $D \cap B = 1$.*

Proof. By Lemma 1.11 there exists such homomorphism $\beta : Q \to Q$, that $\beta\alpha = \varepsilon$ for the natural inclusion $\alpha : D \to Q$ and the identity mapping $\varepsilon : D \to D$. By Lemma 1.10 the subloop D is normal in Q, therefore $Q = D \times \ker\beta$.

Let now the equality $B \cap D = 1$ hold true for the subloop $B \subseteq Q$. We denote $H =< D, B >$. By Lemma 1.10 $D \subseteq Z(Q)$ is the centre of the loop H, then it is easy to show that any element of the CML H has the form au, where $a \in B, u \in D$. By (1.2) and (1.5) we have $L(au, bv)c = c(c, bv, au) = c(c, b, a) \in B$, for any $a, b \in B$ and any $u, v \in D$. Consequently, the subloop B is invariant in regard to the inner mapping group of the CML H, i.e. the subloop B is normal in H. Then $< B, D >= B \times D$ and there is a homomorphism $\xi : B \times D \to D$ coinciding with the identity on D and unitary on B. If we replace ε by ξ in the first part of this proof, then we obtain $Q = D \times \ker \beta$, where $B \subseteq \ker \beta$. This completes the proof of Proposition 1.12.

The second part of this proposition states that a divisible CML is an absolute direct factor.

If the CML Q is given, let us examine the subloop D within it, generated by all divisible subloops of the CML Q. By Lemma 1.10 they all belong to the centre $Z(Q)$ of the CML Q, then it is easy to see that D is a divisible CML. Thus it is the maximal divisible subloop of the CML Q. By Proposition 1.12 $Q = D \times C$, where obviously C is a reduced CML, meaning that it has no non-unitary divisible subloops. Consequently, we obtain

Proposition 1.13. *Any commutative Moufanf loop Q is a direct product of the divisible subloop D that lies in the centre $Z(Q)$ of the loop Q, and the reduced subloop C. The subloop D is unequivocally defined, the subloop C is defined exactly till the isomorphism.*

Proof. Let us prove the last statement. As D is the maximal divisible subloop of the CML Q, it is entirely characteristic in C, i.e. it is invariant in regard to the endomorphisms of the CML Q. Let $Q = D' \times C'$, where D' is a divisible subloop, and C' is a reduced subloop of the CML Q. We denote by φ, ψ the endomorphisms $\varphi : Q \to D', \psi : Q \to C'$. As D is an entirely characteristic subloop, φD and ψD are subloops of the loop Q. It follows from the inclusions $\varphi D \subseteq D'$ and $\psi D \subseteq C'$ that $\varphi D \cap \psi D = 1$. By Lemma 1.10 D is an abelian group, therefore $\varphi D, \psi D$ are normal in D. Then $d = \varphi d \cdot \psi d$ $(d \in D)$ gives $D = \varphi D \cdot \psi D$, so $D = \varphi D \times \psi D$. Obviously, $\varphi D \subseteq D \cap D', \psi D \subseteq D \cap C'$, where from $D = (D \cap D') \times (D \cap C')$. But $D \cap C' = 1$ as a direct factor of the divisible CML, that is contained by the reduced CML. Therefore, $D \cap D' \subseteq D, D \subseteq D'$, i.e. $D = D'$. This completes the proof of Proposition 1.13.

Let us finally prove

Proposition 1.14. *The following conditions are equivalent for the commutative Moufang loop D:*

1) D is a divisible loop;

2) D is an injective loop;

3) D serves as a direct factor for any commutative Moufang loop that contains it.

Proof. The implication 1) $\longrightarrow$ 2) is proved in Lemma 1.11.

2) $\longrightarrow$ 3). By the definition of the injective CML D there is such a homomorphism $\beta : Q \to D$ that $\beta\alpha = \epsilon$ for the natural inclusion $\alpha : D \to Q$ and identity mapping $\epsilon : D \to D$. We denote $\ker\beta = H$. Obviously $Q =< D, H >, H \cap D = 1$ and if $aH = bH$, then $a = b$. Let $x \in Q, d \in D, h \in H$. The CML is an IP-loop, then $(L(x,h)d)H = ((xh)^{-1}(x \cdot hd)H = (x^{-1}(xd))H = dH$, i.e. $L(x,h)d = d$. Any element from Q has the form dh, where $d \in D, h \in H$. Using (1.2) and (1.5) it is easy to show then that the subloop D is invariant in regard to the inner mapping group of the CML Q, i.e. D is normal in Q. Consequently, $Q = D \times H$.

3) $\longrightarrow$ 1). Let the CML D satisfy the condition 3) and let there exist such generators a, b, c of the CML D that $(a, b, c) \neq 1$. Let us examine the CML $Q =< D, x >$, where the element x does not belong to D and given by all the identity relations $(a, u, v) = (x, u, v)$ for any $u, v \in D$. Obviously, D is a subloop of the CML Q, then it serves as a direct factor. Therefore the element x associates with any two elements of the subloop D, in particular, $(x, b, c) = 1$. But $(x, b, c) = (a, b, c) \neq 1$. Contradiction. A consequently, the CML D is associative. By [6] any abelian group can be embedded as a subgroup into a divisible group. Therefore the CML D is divisible. This completes the proof of Proposition 1.14.

2 Finitely cogenerated commutative Moufang loops

A subset H of the CML Q is called *self-conjugate* if $I(Q)H = H$, where $I(Q)$ is the inner mapping group of the CML Q. A self-conjugate set L of elements of the loop Q will be called a *normal system of cogenerators* if any homomorphism $\varphi : Q \to H$ for which $L \cap \ker\varphi \neq \emptyset$ or $\{1\}$ is a monomorphism, for any loop H. Obviously it is equivalent to the fact that any non-unitary normal subloop of the loop Q contains an non-unitary element from L.

A loop Q will be called *finitely cogenerated* if it possesses a finite normal system of cogenerators.

Theorem 2.1. *The following conditions are equivalent for an arbitrary commutative Moufang loop Q:*

1) Q is a finitely cogenerated loop;

2) the loop Q possesses a finite normal subloop B such that $B \cap H \neq \{1\}$, for any normal subloop H of the loop Q;

3) the loop Q is a direct product of a finite number of quasicyclic groups that lie in the centre $Z(Q)$ of the loop Q and a finite loop;

4) the loop Q satisfies the minimum conditions for subloops;

5) the loop Q possesses a finite series of normal subloops any factor of which is either a group of a simple order, or a quasicyclic group.

Proof. 1) $\longrightarrow$ 2). Let L be a finite normal system of cogenerators of the CML Q and $a \in Q$ be an element of an infinite order. By Lemma 1.2 the subloop $< a^{3^n} >$ is normal in the CML Q. The intersection $< a^{3^n} > \cap L$ is either null, or equal to $\{1\}$ for a certain large n, that contradicts the condition 1). Therefore there are no elements of an infinite order in the CML Q. Then, by Lemma 1.3, the subloop

$< L >$ is finite. The system of cogenerators is self-conjugate in the CML Q, then the subloop $< L >$ is normal in Q, as the inner mappings are automorphisms in the CML [4]. Consequently, the condition 2) holds in the CML Q.

2)$\longrightarrow$ 3). It can be shown that the CML Q is periodic, as it was done when proving the implication 1) $\longrightarrow$ 2). Then, by Lemma 1.4, it decomposes into a direct product of its maximal p-subgroups, therefore Q contains a finite number of such p-subloops. In order to prove 3) we can suppose that Q, B are 3-loops.

Like in abelian groups [6] the non-negative number n for which the equality $x^{3^n} = a$ has solutions in Q will be called the 3-*height* $h(a)$ of the element a. If the equality $x^{3^n} = a$ has solutions for any n, then a will be called the *infinite* 3-*height*, $h(a) = \infty$.

We denote $Q[3] = \{x \in Q | x^3 = 1\}$ and let $a \in Q[3]$. Then $< \varphi a | \varphi \in I(Q) >$ is the minimal normal subloop containing the element a, where $I(Q)$ is the inner mapping group of the CML Q. By the condition 2) $a \in B$, and then $Q[3]$ will be a finite subloop. It follows from here that the equality $x^3 = a$ can have not more than a finite number of solutions in CML Q, for a fixed element $a \in Q$. If $h(a) = \infty$, then the solutions $x_1, \ldots, x_k$ cannot have all finite heights, as if the equality $y^{3^n} = a$ holds for the element $y \in Q$, then $y^{3^{n-1}}$ is one of the elements $x_1, \ldots, x_k$.

Let now $a_1 \in Q[3], h(a_1) = \infty$. We denote the solution of an infinite height of the equality $a_1 = x^3$ by a_2, the solution of an infinite height of the equality $a_2 = x^3$ by a_3 and so on. Consequently, we have constructed a quasicyclic group which lies in the centre of the CML Q, by Lemma 1.10, i.e. it is normal in Q. The union D of all quasicyclic groups of the CML Q is a divisible group, therefore by Proposition 1.12 $Q = D \times C, D \subseteq Z(Q)$. The subloop C has no element of an infinite height, as if an element $a \in Q$ of the order 3^n $(n \geq 1)$ has an infinite height, then $a^{3^{n-1}} = a^{3^{-1}}, a^{3^{-1}} \in Q[3]$ and the element $a^{3^{-1}}$ has an infinite height. We have shown that $C[3]$ is a finite subloop. If $a \in C[3], a^{3^n} = 1, a = x^{3^m}$, then $x^{3^{n+m}} = 1, x^{3^{m+n-1}} = x^{3^{-1}}, x^{3^{-1}} \in C[3]$, therefore there is an maximum of heights k of the elements of subloops $C[3]$. But then $(C[3])^{k+1} = 1$, but by Lemma 1.3 the subloop C is finite. The finiteness of the quasicyclic groups of the CML Q number follows from the finiteness of the subloop $D[3]$.

3) $\longrightarrow$ 4). This statement follows from that the fact the quasicyclic groups and the direct product of their finite number satisfy the minimum conditions for subgroups.

4) $\longrightarrow$ 1). The CML Q has no elements of an infinite order, as if a is such an element, then $< a^{3^n} >$ $(n = 1, 2, \ldots)$ is a strictly descending series of the subloops of the CML Q. Then, by Lemma 1.4, Q decomposes into the direct product of a finite number of maximal p-subloops Q_p. The subloop $Q_p[p]$ is normal in Q and it cannot be infinite. In such a case the subloop $\prod_p Q_p[p]$ will be a finite normal system of cogenerators.

The implication 3) $\longrightarrow$ 5) follows from Lemma 1.8.

In order to prove the implication 5) $\longrightarrow$ 3) we should first show that if Q has a finite normal subloop H such that the quotient loop Q/H is a quasicyclic group, then Q has a quasicyclic group of an finite index. First we suppose that the subloop H

is associative. By the definition of the quasicyclic group of the CML Q is generated by the set $\{a_0H, \dots, a_iH, \dots,\}$, where $a^p_{i+1}H = a_iH, a_0 \in H, i = 1, 2, \dots$ We will show that $a_i \in Z_Q(H)$ is the centralizer of the subloop H in Q. If $p = 3$, then if follows from the equality $a^3_{i+1}h = a_i$, where $h \in H$, for $h_1, h_2 \in H$ from (1.3)- (1.5), that $(a_i, h_1, h_2) = (a^3_{i+1}h, h_1, h_2) = 1$, i.e. $a_i \in Z_Q(H)$. If $p \neq 3$, then by (1.3), (1.4) $(u^p, v, w) = (u, v, w)^{\pm 1}$. Then we have $(a_1, h_1, h_2) = (a^p_1, h_1, h_2)^{\pm 1} = (h, h_1, h_2)^{\pm 1} = 1$ from the relations $a^p_1 = h \in H$. Further, if $a_i \in Z_Q(H)$ and $a^p_{i+1} = a_ih$, then $(a_{i+1}, h_1, h_2) = (a^p_{i+1}, h_1, h_2)^{\pm 1} = (a_ih, h_1, h_2) = 1$ by (1.5), i.e. $a_{i+1} \in Z_Q(H)$. Therefore $Q = HZ_Q(H)$. As the intersection $H \cap Z_Q(H)$ is contained in the centre of the CML $Z_Q(H)$, and the quotient loop $Z_Q(H)/(Z_Q(H) \cap H)$ is isomorphic to the quasicyclic group $Q/H = Z_Q(H)H/H$ the CML $Z_Q(H)$ is an infinite abelian group, and it satisfies the minimum condition for subgroups. Then it contains a quasicyclic group of finite index [6]. But by the relation $Q = HZ_Q(H)$, the latter has a finite index in the CML Q.

Let now H be an arbitrary subloop. It is finite, then by Lemma 1.8 its upper central series has the form $1 = Z_0 \subset Z_1 \subset \dots \subset Z_{n-1} \subset Z_n = H$, where $Z_i/Z_{i-1} = Z(H/Z_{i-1})$ or $Z_i = \{a \in H | (a, h_1, \dots, h_{2i} = 1 \forall h_1, \dots, h_{2i}) \in H\}$. (Here $(u_1, \dots, u_{2i-1}, u_{2i}, u_{2i+1}) = ((u_1, \dots, u_{2i-1}), u_{2i}, u_{2i+1}))$. The inner mappings are automorphisms in CML [4], then it follows from the last equality that the subloop Z_i is normal in Q, as the subloop H is normal in Q. Further, if follows from the relations

$$Q/H \cong (Q/Z_{n-1})/(H/Z_{n-1}) = (Q/Z_{n-1})/(Z_n/Z_{n-1}) = (Q/Z_{n-1})/Z(H/Z_{n-1})$$

and according to the previous case that the CML $Q/Z_{n-1})$ contains a quasicyclic group of finite index. Without loss of generalitiy, we will consider that Q/Z_{n-1} is a quasicyclic group, by Proposition 1.14. Let us now suppose that Q/Z_i $(i \leq n-1)$ is a quasicyclic group. Then it follows from the relations $Q/Z_i \cong (Q/Z_{i-1})/(Z_i/Z_{i-1}) = (Q/Z_{i-1})/Z(Q/Z_{i-1})$ that Q/Z_{i-1} is a quasicyclic group. We obtain for $i = 1$ that Q contains a quasicyclic group of finite index.

It is obvious, that the implication 5) $\longrightarrow$ 3) should be proved supposing that the CML Q contains a series of normal subloops

$$1 = H_0 \subset H_1 \subset \dots \subset H_m = Q \qquad (2.1)$$

with $m \geq 2$ that have infinite factors and all are quasicyclic groups.

Let us show that the series (2.1) contains a member which has a quasicyclic group of finite index. If the subloop H_1 is infinite, then the statement is obvious. But if it is finite, then let H_k be such a finite member of the series (2.1) that the next number H_{k+1} is infinite. Then H_{k+1} contains a quasicyclic group L_{k+1} of finite index. If all factors of the series (2.1) which are after the factor H_{k+1}/H_k are finite, then L_{k+1} has a finite index in Q and by Proposition 1.14 the statement 3) holds in the CML Q.

Let H_{n+1}/H_n be the first infinite factor among those that are after H_{k+1}/H_k. By Lemma 1.10 the subloop L_{k+1} is normal in Q. There exists a finite normal subloop

H_n/L_{k+1} in the CML H_{n+1}/L_{k+1} on which the quotient loop is a quasicyclic group. By the above proved, the CML H_{n+1}/L_{k+1} contains a quasicyclic group L_{n+1}/L_{k+1} of finite index. In the CML the quasicyclic groups lie in the centre (Lemma 1.10), then L_{n+1} is a product of two quasicyclic groups. Continuing these reasonings, after a finite number of steps we will obviously obtain that the CML Q contains a subloop that is the direct product of a finite number of quasicyclic groups of finite index. Then the CML Q satisfies the condition 3). This completes the proof of Theorem 2.1.

Corollary 2.2. *The commutative Moufang loops satisfying the minimum condition for subloops, compose a class closed in regard to the extension.*

The statement follows from the equivalence of the conditions 4) and 5) of Theorem 2.1.

Corollary 2.3. *The commutative Moufang loops, satisfying the minimum condition for subloop, are centrally nilpotent.*

The statement follows from the equivalence of the conditions 3), 4) of the Theorem 2.1 and Lemma 1.8.

Corollary 2.4. *The set of elements of any order is finite in the commutative Moufang loop satisfying the minimum condition for subloops.*

3 The multiplicative groups of commutative Moufang loops with minimum condition for subloops

Let Q be an arbitrary CML and let H be a subset of the set Q. Let $\mathbf{M}(H)$ denote a subgroup of the multiplicative group $\mathfrak{M}(Q)$ of the CML Q, generated by the set $\{L(x)|\forall x \in H\}$. Takes place

Lemma 3.1. *Let the commutative Moufang loop Q with the multiplicative group $\mathfrak{M}$, $Z(\mathfrak{M})$, which is the centre of the group $\mathfrak{M}$ and the centre $Z(Q)$ decompose into the direct product $Q = D \times H$, moreover, $D \subseteq Z(Q)$. Then $\mathfrak{M} = \boldsymbol{M}(D) \times \boldsymbol{M}(H)$, and besides, $\boldsymbol{M}(D) \subseteq Z(\mathfrak{M}), \boldsymbol{M}(D) \cong D$.*

Proof. It is obvious that any element $a \in Q$ has the form $a = dh$, where $d \in D, h \in H$. As $d \in Z(Q)$, then $L(a) = L(d)L(h)$, therefore $\mathfrak{M} =< \mathbf{M}(D), \mathbf{M}(H) >$. It follows from the equality

$$Z(\mathfrak{M}) = \{\varphi \in \mathfrak{M}|\varphi = L(a)\forall a \in Z(Q)\}$$

that $\mathbf{M}(D) \subseteq Z(\mathfrak{M})$, therefore it is easy to see that the subgroups $\mathbf{M}(D), \mathbf{M}(H)$ are normal in $\mathfrak{M}$ and $\mathbf{M}(D) \cong D$. Finally, if $\varphi \in \mathbf{M}(D) \cap \mathbf{M}(H)$, then $\varphi = L(u), L(u)1 \in D \cap H, \varphi$ is an inner mapping. Consequently, $\mathfrak{M} = \mathbf{D} \times \mathbf{H}$, as required.

Corollary 3.2. *The multiplicative group $\mathfrak{M}$ of the periodic commutative Moufang loop Q decomposes into the direct product of its maximal p-subgroups $\mathfrak{M}_3$, moreover, $\mathfrak{M}_p \subseteq Z(\mathfrak{M})$ for $p \neq 3$.*

Proof. By Lemma 1.4 the CML Q decomposes into the direct product of its maximal p-subgroups, moreover, $Q_p \subseteq Z(Q)$ for $p \neq 3$. Then it follows from lemma 3.1 that the group $\mathfrak{M}$ decomposes into a direct product of the subgroups $\mathbf{M}(Q_p)$, moreover, $\mathbf{M}(Q_p) \subseteq Z(\mathfrak{M})$ and $\mathbf{M}(Q_p) \cong Q_p$ for $p \neq 3$. In order to finish the proof, it should be shown that $\mathbf{M}(Q_p)$ is a 3-group. But this is shown in the next lemma.

Lemma 3.3. *The multiplicative group $\mathfrak{M}$ of the commutative Moufang 3-loop Q is a 3-group.*

Proof. Let γ be an arbitrary element from $\mathfrak{M}$. Then γ can be presented as a product of a finite number of translation $\gamma = L(u_1)L(u_2)\dots L(u_n)$, where $u_1, u_2, \dots, u_n \in Q$. We denote $L =< u_1, u_2, \dots, u_n >$. For any element $x \in Q$ we denote by $H(x)$ the subloop of CML Q, generated by set $x \cup L$, by $\mathfrak{N}(x)$ – the multiplicative group of CML $H(x)$, and by Γ – the subgroup of group $\mathfrak{M}$ generated by the translations $L(u_i), i = 1, \dots, n$. By Lemmas 1.8 and 1.3 $H(x)$ is a finite centrally nilpotent 3-loop. Let us show that $\mathfrak{N}(x)$ is a 3-loop. Indeed, we denote $H(x) = G$. By Lemma 1.3, Chap. IV from [4] $\mathfrak{M}(Z/Z(G)) \cong \mathfrak{M}(G)/Z^*$, where $Z^* = \{\alpha \in \mathfrak{M}(G) | \alpha x \cdot Z(G) = x \cdot Z(G) \; \forall x \in G\}$. If $\theta \in Z^*$, then we define the function $f : G \longrightarrow Z(G)$ by the rule $\theta x = xf(x)$ for $\forall x \in G$. Obviously, $f(x) \in Z(G)$. If $\eta \in Z^*$ and $\eta x = xg(x)$, then $(\theta\eta)x = \theta(L(g(x))x) = L(g(x))\theta x = (g(x)f(x))x$. Consequently, Z^* is isomorphic to the group of one-to-one mappings of CML Q on $Z(G)$. Therefore Z^* is a 3-group. If CML G is centrally nilpotent of the class k, then $G/Z(G)$ is centrally nilpotent of class $k-1$. Then by inductive assumption $\mathfrak{M}(G)/Z^*$ is a 3-group, therefore $\mathfrak{M}(G)$ is also 3-group.

The restriction Γ on $H(x)$ is a homomorphism of Γ on the subgroup of the group $\mathfrak{N}(x)$ which maps the element $\gamma \in \Gamma$ into the element $L(u_1)\dots L(u_n)$ from $\mathfrak{N}(x)$ of the order 3^t. Moreover, Γ maps $H(x)$ into itself. Consequently, γ^{3^t} induces an identity mapping on $H(x)$. In particular, γ^{3^t} maps x into itself for any x from Q. Therefore γ has the order 3^t. This completes the proof of Lemma 3.3.

Lemma 3.4. *The multiplicative group $\mathfrak{M}$ of an arbitrary commutative Moufang loop is locally nilpotent. But if group $\mathfrak{M}$ is periodic, then it is locally finite.*

The proof of the first statement follows from Lemma 1.1. The second statement follows from the well-known fact of the group theory: a periodic locally nilpotent group is locally finite.

Now we can characterize CML, with the minimum conditions for subloops with the help of their multiplicative groups.

Theorem 3.5. *For an arbitrary non-associative commutative Moufang loop Q with a multiplicative group $\mathfrak{M}$ the following conditions are equivalent:*

1) loop Q satisfies the minimum condition for subloops;

2) group $\mathfrak{M}$ is a product of a finite number of quasicyclic groups lying in the centre of the group $\mathfrak{M}$ and a finite group;

3) group $\mathfrak{M}$ satisfies the minimum condition for subgroup;

4) group $\mathfrak{M}$ satisfies the minimum condition for normal subgroup;

5) group $\mathfrak{M}$ satisfies the minimum condition for non-abelian subgroup;

6) at least one maximal abelian subgroup of the group $\mathfrak{M}$ satisfies the minimum conditions for subgroups;

7) if group $\mathfrak{M}$ contains a solvable subgroup of the class r, then $\mathfrak{M}$ satisfies the minimum condition for solvable subgroups of the class r;

8) if group $\mathfrak{M}$ contains a nilpotent subgroup of the class n, then $\mathfrak{M}$ satisfies the minimum condition for nilpotent subgroups of the class n.

Proof. 1) $\longrightarrow$ 2). If CML Q satisfies the minimum condition for subloops, then by Theorem 2.1 $Q = D \times H$, where H is the direct product of a finite number of quasicyclic groups, besides, $D \subseteq Z(Q)$, and H is a finite CML. Then by Lemma 3.1 $\mathfrak{M} = \mathbf{M}(D) \times \mathbf{M}(H)$, and besides $\mathbf{D} \subseteq Z(\mathfrak{M}), \mathbf{M}(D) \cong D$. The group $\mathbf{M}(H)$ is finitely generated, then by Lemma 3.3 H is finite, as it follows from Corollary 3.2 that a multiplicative group of a periodic CML is periodic.

The implication 2) $\longrightarrow$ 3) is obvious. Let now the group $\mathfrak{M}$ satisfy the condition 3), and the CML Q do not satisfy the condition 1), and let $Q \supset H_1 \supset H_2 \supset \ldots \supset H_i \supset \ldots$ be an infinite descending series of subloops of the CML Q. It is easy to see that $\mathbf{M}(H_i) \neq \mathbf{M}(H_{i+1})$ follows from $H_i \neq H_{i+1}$, using the relation $\mathbf{M}(H_i)1 = H_i$, where $\mathbf{M}(H_i)1 = \{\alpha 1 | \alpha \in \mathbf{M}(H_i)\}$. But it contradicts the condition 3). Consequently, 3) $\longrightarrow$ 1).

By Lemma 3.4 the group $\mathfrak{M}$ is locally nilpotent, then the implications 3) $\longleftrightarrow$ 4), 3) $\longleftrightarrow$ 5) follow, respectively, from Theorems 1.24 and Corollary 6.2 from [7].

6) $\longrightarrow$ 3). Let the maximal abelian subgroup $\mathfrak{N}$ of the group $\mathfrak{M}$ satisfy the minimum condition for subgroups. By Lemma 1.1 the quotient group $\mathfrak{M}/Z(\mathfrak{M})$ is a 3-group, therefore by the periodicity of $\mathfrak{N}$, the group $\mathfrak{M}$ is also periodic. Thereof, and in view of Corollary 3.2, we will consider $\mathfrak{M}$ a 3-group. By Lemma 3.4 the group $\mathfrak{M}$ is locally nilpotent. Then the condition 6) $\longrightarrow$ 3) follows from the statement that is proved using Lemma 1.6, analogous to Theorem 1.19 from [7]:

if at least one maximal abelian subgroup of the locally nilpotent p-group satisfies the minimum condition for subgroup, then the group satisfies this condition itself.

By Lemma 3.4 the group $\mathfrak{M}$ is locally nilpotent. It is proved in [8] that for such groups the conditions 3), 7), 8) are equivalent.

Finally, the implication 3) $\longrightarrow$ 6) is obvious. This completes the proof of Theorem 3.5.

It is proved in [7] that if the locally finite p-group has a finite maximal elementary abelian subgroup (respect., a finite set of elements of any order different from unitary element), then it satisfces the minimum condition for subgroups (Theorem 1.21 (respect., Theorem 3.2)). Then from Lemmas 3.3, 3.4 and Theorem 3.6 follows the truth of the following statement.

Proposition 3.6. *The following conditions are equivalent for an arbitrary commutative Moufang 3-loop with a multiplicative group $\mathfrak{M}$:*

1) the loop Q satisfies the minimum condition for subloops;

2) the group $\mathfrak{M}$ contains only a finite set of elements of a certain order different from the unitary element.

Finally, let us prove the statement.

Proposition 3.7. *The following conditions are equivalent for an arbitrary nonassociative commutative Moufang ZA-loop Q with a multiplicative group $\mathfrak{M}$:*

1) the loop Q satisfies the minimum condition for subloops;

2) the group $\mathfrak{M}$ satisfies the minimum condition for noninvariant abelian subgroups.

Proof. Let us first observe that from Lemma 11.4, Chap. VIII from [4] it follows that CML Q is a ZA-loop if and only if its multiplicative group is a ZA-group.

Let us suppose that the group $\mathfrak{M}$ satisfies the minimum condition for noninvariant abelian subgroups. It follows from the above-mentioned that it is a ZA-group. If $\mathfrak{M}$ does not contain noninvariant abelian subgroups, then, obviously, each subgroup is normal in it, i.e. it is hamiltonian. However, it is impossible that the multiplicative group of an arbitrary CML cannot contain a nonabelian gamiltonian subgroup. Indeed, arbitrary hamiltonian groups are described by the next theorem [7]:

A hamiltonian group can be decomposed into a direct product of the group of quaternions and abelian groups whose each element's order is not greater than 2. Conversely, a group that has such a decomposition is hamiltonian.

A *group of quaternions* is the group generated by the generators a, b and that satisfies the identical relations $a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1}$. Then it follows from Corollary 3.2 that in the case of a multiplicative group $a = b = 1$. Consequently, the arbitrary hamiltonian group of the multiplicative group of CML is abelian.

Let now $\mathfrak{N}$ be a noninvariant abelian subgroup of the group $\mathfrak{M}$ and α be an element of infinite order from $\mathfrak{M}$. By Lemma 1.1 the quotient group $\mathfrak{M}/Z(\mathfrak{M})$ is a 3-group, therefore $\alpha^{3^k} \in Z(\mathfrak{M})$ for a certain natural number k. This means that the descending series of noninvariant associative subgroups

$$< \mathfrak{N}, \alpha^{3^k} >\supset< \mathfrak{N}, \alpha^{3^{k+1}} >\supset \ldots \supset< \mathfrak{N}, \alpha^{3^{k+i}} >\supset \ldots$$

of the group $\mathfrak{M}$ does not break. But it contradicts the condition 5). Consequently, the group $\mathfrak{M}$ is periodic. In such a case, we will consider by Corollary 3.2 that $\mathfrak{M}$ is a 3-group.

Let us suppose that the group $\mathfrak{M}$ does not satisfy the minimum condition for subgroups. Then, by Lemma 3.4 and Theorem 1.21 from [7] the group $\mathfrak{M}$ contains the infinite direct product

$$\mathfrak{N} = \mathfrak{N}_1 \times \mathfrak{N}_2 \times \ldots \times \mathfrak{N}_n \times \ldots$$

of cyclic groups of the order three. If α is an arbitrary element from the centralizer $Z_{\mathfrak{M}}(\mathfrak{N})$ of the subgroup $\mathfrak{N}$ in $\mathfrak{M}$, then there exists such a number $n = n(\alpha)$ that

$$< \alpha > \cap(\mathfrak{N}_{n+1} \times \mathfrak{N}_{n+2} \times \ldots) = 1.$$

As the group $\mathfrak{M}$ satisfies the minimum condition for noninvariant abelian subgroups, the infinite descending series of abelian subgroups

$$\Re^k(\alpha) \supset \Re^{k+1}(\alpha) \supset \ldots,$$

where $\Re^k(\alpha) =< \alpha > (\mathfrak{N}_{k+1} \times \mathfrak{N}_{k+1} \times \ldots)$, contains an noninvariant subgroup $\Re^k(\alpha)$ $(r = r(\alpha))$, beginning with a certain natural number $k \geq n$. As the intersection of all such noninvariant subgroups coincides with the subgroup $< \alpha >$, the latter is normal in $\mathfrak{M}$. But α is an arbitrary element from the centralizer $Z_{\mathfrak{M}}(\mathfrak{N})$, and it means that $Z_{\mathfrak{M}}(\mathfrak{N})$ is a hamiltonian group. From here follows that $Z_{\mathfrak{M}}(\mathfrak{N})$ is an abelian group. Obviously, $\mathfrak{N}_i \subseteq Z_{\mathfrak{M}}(\mathfrak{N})$, then the minimal subgroup $\mathfrak{N}_i$ is normal in $\mathfrak{M}$. By Proposition 1. 6 from [7], in a ZA-group the minimal normal subgroups are contained in its centre. Then $\mathfrak{N}_i \subseteq Z_{\mathfrak{M}}(\mathfrak{N})$, therefore $Z_{\mathfrak{M}}(\mathfrak{N}) = \mathfrak{M}$. As $Z_{\mathfrak{M}}(\mathfrak{N})$ is an ablian group, the last equality contradicts the fact that $\mathfrak{M}$ is an noninvariant group. Consequently, the group $\mathfrak{M}$ satisfies the minimum condition for subgroups. Then the equivalence of the conditions 1) and 2) follows from the Theorem 3.5.

4 The commutative Moufang loops with the minimum condition for normal subloops

If it does not cause any misunderstandings, we will further omit the words "for subloops" in the expression "minimum condition for subloops".

Lemma 4.1. *Let the series*

$$1 = Z_0 \subset Z_1 \subset \ldots \subset Z_\alpha \subset \ldots \subset Z_\beta \subset \ldots \subset Z_\gamma = Q \tag{4.1}$$

be the upper central series of the commutative Moufang ZA-loop Q, H be its arbitrary normal subloop. Then the non-emptiness of the intersection $H \cap (Z_\beta \backslash Z_\alpha$ follows from the non-emptiness of the intersection $H \cap (Z_{\beta+1} \backslash Z_\beta$ for any $\beta > \alpha$.

Proof. Let $h \in H \cap (Z_{\beta+1} \backslash Z_\beta)$. The existence of such elements $a, b \in Q$ that $(h, a, b) \in H \cap (Z_\beta \backslash Z_\alpha)$ follows from the normality of the subloop H and the definition of the members of series (4.1). Indeed, if $(h, a, b) \in Z_\alpha$ for all $a, b \in Q$, then $h \in Z_{\alpha+1} \subset Z_\beta$. So, $h \notin Z_{\beta+1} \backslash Z_\beta$, and it contradicts the choice of the element n. This completes the proof of Lemma 4.1.

Lemma 4.2. *Let the commutative Moufang ZA-loop Q be the finite extension of the loop Q satisfies the minimum condition if and only if the centre $Z(H)$ of the loop H also satisfies this condition.*

Proof. Let us suppose that the centre $Z(Q)$ satisfies the minimum condition for subloops, and let $a_1, \ldots, a_n$ be representations of cosets of Q modulo H, taken by one from each coset. We denote $L =< Z(H), a_1, \ldots, a_n >$. Let us show that the centre $Z(L)$ of the CML L satisfies the minimum condition. Indeed, the intersection $Z(L) \cap Z(H)$ is contained into $Z(Q)$, therefore it is a group with minimum condition. Obviously, the index $Z(H)$ in $< Z(H), Z(L) >$ is finite. We have

$$< Z(H), Z(L) > /Z(H) \cong Z(L)/(Z(L) \cap Z(H)).$$

It follows from this relation that $Z(L)$ which is a finite extension of the group $Z(H) \cap Z(L)$ satisfies the minimum condition, satisfies this condition itself.

Let N_k be a subgroup of the group $Z(L)$ generated by all its elements whose orders are divisible by p^k. The group N_k is finite, as the group $Z(L)$ satisfies the minimum condition. We denote by Z_k the subgroup of the group $Z(H)$ generated by all its elements whose orders are divisors of p^k. If $Z(H)$ does not satisfy the minimum condition, then Z_k should be infinite. Let

$$Z_k = Z_k^{(1)} \times \ldots \times Z_k^{(m)} \times Z_k^{(m+1)} \times \ldots$$

be the decomposition of the group Z_k into an infinite direct product of cyclic groups. If the intersection $Z_k \cap N_k$ is contained into the finite direct product $Z_k^{(1)} \times \ldots \times Z_k^{(m)}$, then the intersection of the groups $M_k = Z_k^{(m+1)} \ldots$ and $Z(L)$ should contain only the unitary element:

$$M_k \cap Z(L) = 1. \tag{4.2}$$

By Lemma 1.1 the subgroup Φ of the inner mapping group of the CML Q generated by all the mappings of the form $L(a_i, a_j)$, $i, j = 1, \ldots, n$, is finite. The subgroups $\varphi M_k, \varphi \in \Phi$, is a (finite) the set of all conjugated subloops with M_k in the CML Q, because the elements $a_1, \ldots, a_n$ present a full system of representations of cosets of CML Q modulo H, and $M_k \subseteq Z(H)$. The intersection

$$R_k = \cap_{\varphi \in \Phi} \varphi M_k$$

is obviously an infinite normal subloop in Q. We remind that $R_k \subseteq L$, as $M_k \subseteq Z(L), \varphi M_k \subseteq L$. By Lemma 1.8 and Lemma 1.6 $R_k \cap Z(L) \neq 1$, that contradicts (4.2). Consequently, the assumption that $Z(H)$ does not satisfy the minimum condition is not true.

Conversely, let $Z(Q)$ does not satisfy the minimum condition. As H has a finite index in Q, then it follows from the relation

$$Z(Q)H/H \cong Z(Q)/(Z(Q) \cap H)$$

that $Z(Q) \cap H$ has a finite index in $Z(Q)$. Consequently, $Z(Q) \cap H$ does not satisfy the minimum condition. But $Z(Q) \cap H \subseteq Z(H)$, therefore $Z(H)$ does not satisfy the minimum condition as well. This completes the proof of Lemma 4.2.

Lemma 4.3. *If the commutative Moufang ZA-loop, which is a finite extension of the loop H, possesses a normal subloop K, which lies in the centre $Z(H)$ of the loop H and does not satisfy the minimum condition, then the intersection of H with the centre $Z(Q)$ of the loop Q does not satisfy the minimum condition as well.*

Proof. By Lemma 1.4 we'll consider that the CML Q is a 3-loop. We denote by L the lower layer of the abelian group K. As K does not satisfy the minimum condition, L is infinite.

Let us first examine the case when the quotient loop Q/H is associative. Let

$$1 = g_1, g_2, \ldots, g_n$$

be a full system of representations of cosets of CML Q modulo H. We suppose by inductive considerations that the intersection of L_{i-1} of the centre of the CML $< H, g_1, \ldots, g_{i-1} >$ with the subloop L is infinite. As the quotient loop Q/H is associative, the subloop $< H, g_1, \dots, g_{i-1} >$ inverse image of a normal subloop under the homomorphism $Q \longrightarrow Q/H$, is normal in Q. The subloop L is invariant in regard to all automorphisms of the normal subloops H of the CML Q. In the CML the inner mappings are its automorphisms [4]. Then the subloop L is invariant in regard to the inner mapping group of the CML Q, i.e., it is normal in Q. Therefore the intersection L_{i-1} is also a normal subloop in Q. Let us examine the CML $< L_{i-1}, g_i >$. By Lemma 4.2 this loop's center does not satisfy the minimum condition. Consequently, if the order of the element g_i is 3^k, then there exists such a number $r \leq 3^k$ that for the infinite set of elements P of the order 3 from the CML L_{i-1}, the elements of the form $pg_i^r, p \in P$, belong to the centre of the CML $< L_{i-1}, g_i >$. Now, with the help of (1.1) we obtain for $p, q \in P$

$$g_i(g_i^r p \cdot g_i^r q) = (g_i \cdot g_i^r p)(g_i^r q),$$

$$g_i(g_i^{2r} \cdot pq) = (g_i^r \cdot g_i p)(g_i^r q).$$

$$g_i^{2r}(g_i \cdot pq) = g_i^{2r}(g_i p \cdot q),$$

$$g_i \cdot pq = g_i p \cdot q.$$

The last equality shows that the infinite CML $P_i =< P >$ of the index three belongs to the centre of the CML $< H, g_1, \dots, g_{i-1} >$. As L_{i-1} belongs to the centre $< H, g_1, \dots, g_{i-1} >$ and $P_i \subseteq P_{i-1}$, the CML P_i belongs to the centre $< H, g_1, \dots, g_{i-1}, g_i >$. So, the intersection of this CML's centre with L_{i-1} is infinite, therefore it does not satisfy the minimum condition. But $L_{i-1} \subseteq L_i \subseteq H$, then the statement is proved in this case.

Let now Q/H be an arbitrary finite CML and by Lemma 1.8 let

$$\overline{1} \subset Z_1/H \subset \ldots \subset Z_k/H = \overline{Q}$$

be the upper central series of the CML Q/H. By the first case, the intersection of the centre of the CML Z_1 with the subloop L is infinite. As it has already been proved that the intersection of the centre of the CML Z_i with the subloop L is infinite, then applying the first case's results to the CML Z_i and Z_{i+1} we obtain that the intersection of the centre of the CML Z_{i+1} with the subloop L is also infinite. For $i + 1 = k$ follows the lemma's statement.

Lemma 4.4. *If the periodic commutative Moufang ZA-loop contains an associative normal subloop H that does not satisfy the minimum condition, then the latter contains a normal subloop of the loop Q different from itself that does not satisfy the minimum condition as well.*

Poof. By Lemma 1.4 we will consider that Q is a 3-loop. We denote by L the lower layer of the group H. As H does not satisfy the minimum condition, L is infinite. Let

$$1 \subset Z_1 \subset \ldots \subset Z_\gamma = Q$$

be the upper central series of the CML Q. If $L \subseteq Z_1$, then the lemma is proved.

Let us suppose that L does not belong to Z_1. The product $LZ_1 = Q_1$ does not satisfy the minimum condition. The subloop L is contained in the centre of the CML Q_1 (Q_1 is associative). By the assumption L does not belong to the centre of the CML Q, so, there exists such an ordinal number α less than γ that the centre of the CML $Z_\alpha L = Q_{\alpha+1}$ does not contain L in its centre anymore. Consequently, there is such an element a in $Z_{\alpha+1}$ that the centre C of the finite extension $< Q_\alpha, a >$ of the CML Q does not contain the subloop L. By Lemma 4.2 the centre C does not satisfy the minimum condition. The normality of the subloop $< Z_\alpha, a >$ in the CML Q follows from the relation $Z_{\alpha+1}/Z_\alpha = Z(Q/Z_\alpha)$, and hereof follows the normality of the subloop $< Q_\alpha, a >$. Consequently, the centre C of the subloop $< Q_\alpha, a >$ is normal in the CML Q. By Lemma 4.3 the intersection $C \cap L$ does not satisfy the minimum condition. It is different from the subloop L, as the latter does not belong to C. As this intersection is normal in Q, the statement is proved.

Corollary 4.5. *In the periodic commutative Moufang ZA-loop Q each associative normal subloop which satisfies the minimum condition for the normal subloops of the loop Q satisfies the minimum condition for its subloops.*

This statement follows from Lemma 4.4.

Theorem 4.6. *If at least one maximal associative subloop of the commutative Moufang ZA-loop Q satisfies the minimum condition for the normal subloops of the loop Q, then Q satisfies the minimum condition for subloops.*

Proof. By Lemma 1.2 we will consider that the CML Q is periodic. Then the statement follows from Corollary 4.5 and Lemma 1.9.

Corollary 4.7. *In the commutative Moufang ZA-loop the minimum condition for subloops and associative normal subloops are equivalent.*

Corollary 4.8. *If in a commutative Moufang ZA-loop at least one maximal associative normal subloop is finite, then the loop Q is also finite.*

The statement follows from the Theorems 4.6 and 2.1.

Corollary 4.9. *The infinite commutative Moufang ZA-loop Q has an infinite centre.*

Proof. By Corollary 4.8 the CML Q possesses an infinite associative normal subloop. Then the statement follows from Lemma 1.7.

We remark that in [4] an example of a CML with unitary centre is constructed.

Theorem 4.10. *If the centre $Z(Q)$ of the commutative Moufang ZA-loop Q satisfies the minimum condition for subloops, then the loop Q satisfies the minimum condition for subloop itself.*

Proof. By Theorem 2.1 the centre $Z(Q)$ decomposes into the direct product of a finite number of quasicyclic groups D and a finite group C, and by Proposition 1.12 $Q = D \times L$. Obviously, the centre $Z(L)$ of the CML L coincides with C. As C is a finite group, then by Corollary 4.9 the CML L is finite. Then the CML Q satisfies the minimum condition for subloops.

Theorem 4.11. *If a commutative Moufang loop satisfies the minimum condition for normal subloops, it satisfies the minimum condition for subloops as well.*

Proof. By Lemma 1.5 an arbitrary CML possesses a central system. It follows from the minimum condition for normal subloops that each central system of the CML Q is an ascending central series, i.e. Q is a ZA-loop. Now the statement follows from Corollary 4.7.

References

[1] Sandu N.I., *On locally normal loops.* – Scripta scientiarum mathematicarum, 1999, **1**, no. 2, p. 364–380.

[2] Sandu N.I., *Commutative Moufang loops with finite classes of conjugate elements.* – Scripta scientiarum mathematicarum, 1999, **1**, no. 2, p. 381–394.

[3] Sandu N.I., *Commutative Moufang loops with finite classes of conjugate subloops.* – Mat. zametki, 2003, **73**, no. 2, p. 269–280 (In Russian).

[4] Bruck R.H., *A survey of binary systems.* – Springer Verlag, Berlin-Heidelberg, 1958.

[5] Sandu N.I., *Centrally nilpotent commutative Moufang loops.* – Quasigroups and loops, Mat. issled., 1979, **51**, p. 145–155 (In Russian).

[6] Fuchs L., *Infinite abelian groups, vol. 1.* – Mir, Moscow, Mir, 1974 (In Russian).

[7] Chernikov S.N., *The groups with given properties of the systems of subgroups.* – Moscow, Nauka, 1980 (In Russian).

[8] Zaitzev D.I., *Steadily solvable and steadily nilpotent groups.* – DAN SSSR, 1967, **176**, no. 3, p. 509–511 (In Russian).

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