On some Hypergroups and their Hyperlattice Structures

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Abstract. Let $G$ be a hypergroup and $\mathcal{L}(G)$ be the set of all subhypergroups of $G$. In this survey article, we introduce some hypergroups $G$ from combinatorial structures and study the structure of the set $\mathcal{L}(G)$. We prove that in some cases $\mathcal{L}(G)$ has a lattice or hyperlattice structure.


Keywords and phrases: Hypergroup, hyperlattice, integer partition.

1 Introduction

First of all we will recall some algebraic definitions used in the paper. A hyperstructure is a set $H$ together with a function $\cdot : H \times H \longrightarrow P^\star(H)$ called hyperoperation, where $P^\star(H)$ denotes the set of all non-empty subsets of $H$. F.Marty [18] defined a hypergroup as a hyperstructure $(H, \cdot)$ such that the following axioms hold: (i) $(x.y).z = x.(y.z)$ for all $x, y, z$ in $H$, (ii) $x.H = H.x = H$ for all $x$ in $H$. The axiom (ii) is called the reproduction axiom. A commutative hypergroup $(H, o)$ is called a join space if for all $a, b, c, d \in H$, the implication $a/b \cap c/d \neq \emptyset \implies aod \cap boc \neq \emptyset$ is valid, in which $a/b = \{x \mid a \in xob\}$.

The concept of an $H_v$-group is introduced by T.Vougiouklis in [20] and it is a hyperstructure $(H, \cdot)$ such that (i) $(x.y).z \cap x.(y.z) \neq \emptyset$, for all $x, y, z$ in $H$, (ii) $x.H = H.x = H$ for all $x$ in $H$. The first axiom is called weak associativity.

Following Gionfriddo [12] and Vougiouklis [20], we define a generalized permutation on a non-empty set $X$ as a map $f : X \longrightarrow P^\star(X)$ such that the reproductive axiom is valid, i.e. $\cup_{x \in X} f(x) = f(X) = X$. The set of all generalized permutations on $X$ is denoted by $M_X$. We now assume that $(G, \cdot)$ is a hypergroup and $X$ is a set. The map $\odot : G \times X \longrightarrow P(X)^*$ is called a generalized action of $G$ on $X$ if the following axioms hold:

1) For all $g, h \in G$ and $x \in X$, $(gh) \odot x \subseteq g \odot (h \odot x)$,
2) For all $g \in G$, $g \odot X = X$.

Here, for any $g \in G$ and $Y \subseteq X$, $g \odot Y$ is defined as $\cup_{x \in Y} g \odot x$, and for any $x \in X$ and $B \subseteq G$, $B \odot x$ is, by definition, equal to $\cup_{b \in B} b \odot x$. If the equality holds in the axiom 1) of definition, the generalized action is called strong (see [17]).

Following Konstantinidou and Mittas [15], we define a hyperlattice as a set $H$ on which a hyperoperation $\vee$ and an operation $\wedge$ are defined which satisfy the following...
axioms:
1. \( a \in a \lor a \) and \( a \land a = a \),
2. \( a \lor b = b \lor a \) and \( a \land b = b \land a \),
3. \( (a \lor b) \lor c = a \lor (b \lor c) \) and \( (a \land b) \land c = a \land (b \land c) \),
4. \( a \in [a \lor (a \land b)] \land [a \land (a \lor b)] \),
5. \( a \in a \lor b \) implies that \( b = a \land b \).

It is well known [8] that in a lattice the distributivity of the meet (\( \land \)) with respect to the join (\( \lor \)) implies the distributivity of the join with respect to the meet and vice versa, the lattice is then called distributive. But in a hyperlattice a distinction of several types of distributivity is needed. According to Konstantinidou [16], a hyperlattice \( (H, \lor, \land) \) will be called distributive if and only if,
\[
(a \land (b \lor c)) = (a \land b) \lor (a \land c),
\]
for all \( a, b, c \in H \). Also, the hyperlattice \( (H, \lor, \land) \) is called modular if \( a \leq b \) implies that \( a \lor (b \land c) = b \land (a \lor c) \), for all \( c \in H \).

The second author in [2, 3] and [5], studied the construction of join spaces from some combinatorial structures. In [4], he found a new closed formula for the partition function \( p(n) \). We encourage reader to consult these papers for discussion and background material.

Our notation is standard and taken mainly from [1, 8–10] and [20].

2 The Structure of some Hypergroups

Let \( G \) be a group, \( Sym(G) \) be the group of all permutations on \( G \) and \( Sym_e(G) \) be the stabilizer of the identity \( e \in G \) in \( Sym(G) \). Given two permutations \( \phi \) and \( \psi \) from \( Sym_e(G) \) and an element \( g \in G \), we define a new permutation \( \phi \odot_g \psi = L_{\phi(g)^{-1}} \phi L_g \psi \), where \( L_{\phi(g)^{-1}}, L_g \in Sym(G) \) are left multiplications by the elements \( \phi(g)^{-1} \) and \( g \), respectively.

According to [13], a subgroup \( H \) of \( Sym_e(G) \) closed under taking products of this form is called rotary closed, i.e. \( H \leq Sym_e(G) \) is called rotary closed provided that \( \phi \odot_g \psi \in H \), for all \( \phi, \psi \in H \) and \( g \in G \). A nice family of rotary closed subgroups of \( Sym_e(G) \), for finite \( G \)'s, comes from the theory of Cayley graphs and can be obtained in the following way. Let \( \Omega \) be a set of generators for a finite group \( G \) not containing the identity element \( e \) but containing \( x^{-1} \) together with every \( x \) contained in \( \Omega \). The subgroup \( Rot_\Omega(G) \) of \( Sym_e(G) \) of all permutations preserving \( e \) and satisfying the condition \( \phi(a)^{-1} \phi(ax) \in \Omega \), for every \( a \in G \) and \( x \in \Omega \), is rotary closed (for details see [13] and [14]).

In this section, first we introduce a hyperoperation \( \odot \) on \( Sym_e(G) \) and prove that \( (Sym_e(G), \odot) \) is a hypergroup. Next, we characterize the sub-hypergroups of this hypergroup. To do this, assume that \( \phi, \psi \in Sym_e(G) \), we define \( \phi \odot \psi = \{ \phi \odot_g \psi \mid g \in G \} \).
Proposition 2.1. \((Sym_e(G), \odot)\) is a hypergroup.

Proof. Suppose \(\phi, \psi\) and \(\eta\) are arbitrary permutations of \(Sym_e(G)\). Then we have

\[(\phi \odot \psi) \odot \eta = \{\phi \odot g \psi | g \in G\} \odot \eta = \bigcup_{g \in G} (\phi \odot g \psi) \odot \eta =
\]

\[= \bigcup_{g \in G} \{(\phi \odot g \psi) \odot h \eta | h \in G\} = \{(\phi \odot g \psi) \odot h \eta | g, h \in G\}.
\]

Using similar argument as in above, we can show that

\[\phi \odot (\psi \odot \eta) = \{\phi \odot (g \odot \eta) | g, h \in G\}.
\]

We now assume that \(g, h \in G\), then we have

\[(\phi \odot g \psi) \odot h \eta = L_{\phi \odot g \psi (h)}^{-1} \phi \odot g \psi L_h \eta = L_{\phi (g \psi (h))}^{-1} \phi (g \psi \psi L_h \eta =
\]

\[= L_{\phi (g \psi (h))}^{-1} \phi L_g \psi L_h \eta = L_{\phi (g \psi (h))}^{-1} \phi L_g \psi L_h \eta =\]

\[= L_{\phi (g \psi (h))}^{-1} \phi L_g \psi L_h \eta \psi \eta \eta = \phi \odot (g \psi (h)) \psi \eta \eta \eta \in \phi \odot (\psi \odot \eta).
\]

Therefore, \(\phi \odot (\psi \odot \eta) \subseteq (\phi \odot \psi) \odot \eta\). Using similar argument we have \(\phi \odot (\psi \odot \eta) \subseteq (\phi \odot \psi) \odot \eta\) and the associativity is valid. Next we assume that \(\phi \in Sym_e(G)\) and we have

\[\phi \odot Sym_e(G) = \bigcup_{\psi \in Sym_e(G)} \phi \odot \psi = \bigcup_{\psi \in Sym_e(G)} \{\phi \odot g \psi | g \in G\}.
\]

Suppose \(\delta \in Sym_e(G)\) is arbitrary and \(\psi = L_g^{-1} \phi^{-1} L_{\delta \phi} \delta\). Then, \(\phi \odot g \psi = \delta\) and so \(Sym_e(G) = \phi \odot Sym_e(G)\). Similarly, \(Sym_e(G) \odot \phi = Sym_e(G)\), which completes the proof. \(\Box\)

In what follows, we characterize the sub-hypergroups of the hypergroup \((Sym_e(G), \odot)\).

Proposition 2.2. Let \(G\) be a finite group and \(H\) be a non-empty subset of \(Sym_e(G)\). \(H\) is a sub-hypergroup of \(Sym_e(G)\) if and only if \(H\) is a rotary closed subgroup of \(Sym_e(G)\).

Proof. \((\Rightarrow)\) Suppose \(H\) is a sub-hypergroup of \(Sym_e(G)\). We first show that \(H\) is a closed subset of \(Sym_e(G)\). To do this, suppose \(\phi\) and \(\psi\) are elements of \(H\). Then \(\phi \psi = \phi \odot \psi \psi \subseteq H\) and so \(\phi \psi \in H\). Next, for \(\phi, \psi \in H\) and \(g \in G\), \(\phi \odot g \psi \in \phi \odot \psi \subseteq H\), as desired.

\((\Leftarrow)\) Suppose \(H \leq Sym_e(G)\) is rotary closed and \(\phi \in G\). Since \(H\) is rotary closed \(\phi \odot H \subseteq H\). Suppose \(\psi \in H\). Put \(\eta = \phi^{-1} \psi\) and \(g = e\). Then, \(\phi \odot g \eta = \phi \odot \phi^{-1} \psi = \psi \in \phi \odot \eta\) and so \(H = \phi \odot H\). Similar argument shows that \(H \odot \phi = H\), proving the result. \(\Box\)

It is a well-known fact that the set of all subgroups of a group \(G\) has a lattice structure under the ordinary operations of meet and join. In general, it is far from
true that the set of all sub-hypergroups of a hypergroup has a lattice structure under these operations. In fact, the intersection of two sub-hypergroups of a hypergroup is not necessarily non-empty.

Let \( L(G) \) be the set of all sub-hypergroups of the hypergroup \( G \). In what follows, we show that \( \text{Sym}_e(G) \) has a lattice structure under the ordinary operations of join and meet.

**Proposition 2.3.** \( L(\text{Sym}_e(G)) \) has a lattice structure under the ordinary operations of meet and join.

**Proof.** It is an easy fact that \( \{1_G\} \) and \( \text{Sym}_e(G) \) are rotary closed. Suppose that \( H \) and \( K \) are two rotary closed subgroups of \( \text{Sym}_e(G) \). It is clear that \( H \cap K \) is rotary closed. We claim that \( \langle H, K \rangle \) is also rotary closed. To do this, we assume that \( \psi \in H, \phi \in K \) and \( g \in G \). Then we have:

\[
\psi \circ_g \phi = L_{\psi(g)}^{-1} \psi L_g \phi = L_{\psi(g)}^{-1} \psi L_g \psi \psi^{-1} \phi = \psi \circ_g \psi \psi^{-1} \phi \in \langle H, K \rangle.
\]

Also, for \( \psi_1, \psi_2 \in H, \phi_1, \phi_2 \in K \) and \( g \in G \), we have

\[
\psi_1 \phi_1 \circ_g \psi_2 \phi_2 = L_{\psi_1 \phi_1(g)}^{-1} \psi_1 \phi_1 L_g \psi_2 \phi_2 =
\]

\[
= L_{(\psi_1(\phi_1(g)))^{-1}} \psi_1 L_{\phi_1(g)} \psi_1^{-1} L_{\phi_1(g)}^{-1} \phi_1 L_g \psi_2 \phi_2 =
\]

\[
= (\psi_1 \circ_{\phi_1(g)} \phi_1) \psi_1^{-1} (\phi_1 \circ_g \psi_2 \phi_2) \in HK \subseteq \langle H, K \rangle.
\]

Using similar argument as in above, we can show that \( \langle H, K \rangle \) is a rotary closed subgroup of \( \text{Sym}_e(G) \). This shows that \( L(\text{Sym}_e(G)) \) has a lattice structure under ordinary operations of join and meet. \( \Box \)

Let \( G \) be a set, \( B \) an algebraic Boolean algebra and \( s \) a function from \( G \) into \( B \). We define the hyperoperation \( \circ \) as follows:

\[ a \circ b = \{ x \in G \mid s(x) \leq s(a) \lor s(b) \}. \]

Since for all \( x, y \in G \), \( \{x, y\} \subseteq x \circ y \), \( (G, \circ) \) is an \( H_v \)-group. It is also clear that the hyperoperation \( \circ \) is commutative.

In what follows, we study the sub-hypergroup structure of the hypergroup \( (G, \circ) \). In some special cases we will show that the set \( L(G) \) has a hyperlattice structure. We also assume that \( G_a = \{ g \in G \mid s(g) \leq a \} \). It is easy to see that if \( a \in B \) and \( G_a \neq \emptyset \) then \( G_a \) is an \( H_v \)-subgroup of \( G \). In what follows, when we write \( G_a \), we assume that \( G_a \neq \emptyset \).
**Proposition 2.4.** Let $B$ be a complete Boolean algebra and $s : G \rightarrow B$ be a function such that $(G, *)$ constitute a hypergroup. Also, we assume that that

$$a_1 * a_2 * \cdots * a_n = \{ g \in G \mid s(g) \leq s(a_1) \lor \cdots \lor s(a_n) \},$$

and $H$ is a sub-hypergroup of $G$. Then there exists an element $a \in B$ such that $H = G_a$.

**Proof.** Let $H$ be a sub-hypergroup of $G$ and $a = \lor_{b \in H} s(b)$. We claim that $H = G_a$. To see this, assume $x \in H$. Then $s(x) \leq \lor_{b \in H} s(b) = a$ and so $x \in G_a$, i.e., $H \subseteq G_a$. We now assume that $x \in G_a$. Then $s(x) \leq a = \lor_{b \in H} s(b)$. Since $B$ is algebraic, there are the elements $b_1, b_2, \cdots, b_r$ of $H$ such that $s(x) \leq s(b_1) \lor \cdots \lor s(b_r)$. Now by assumption, $x \in \{ g \in G \mid s(g) \leq s(b_1) \lor \cdots \lor s(b_r) \} = b_1 * b_2 * \cdots * b_r$ and $H$ is a sub-hypergroup of $G$, so $x \in H$, proving the result. \( \square \)

It is clear that $G_{a \lor b} = G_a \cap G_b$, for all $a, b \in B$. It is far from true that $G_{a \lor b} = G_a \cup G_b$. To see this, we construct an example as follows:

**Example 2.5.** Suppose $G = B = P(X)$, $s$ is the identity function, $|X| \geq 3$ and $a, b, c$ distinct elements of $X$. Set $R = \{ a, b \}$ and $S = \{ c \}$. Then $G_R = P(R), G_S = P(S)$ and $G_{R \cup S} = P(R \cup S)$. Now it is easy to see that $G_{R \cup S} \neq G_R \cup G_S$. \( \square \)

By the results of [3] and [4], if the image of $G$ is a $\lor$-sub-semilattice or constitutes a partition of 1, then $\mathcal{L}(G) = \{ G_a \mid a \in B \land G_a \neq \emptyset \}$. In this case, we define a hyperoperation $\lor$ on $\mathcal{L}(G)$ such that $(\mathcal{L}(G), \lor, \land)$ constitutes a hyperlattice. To do this, we assume that $G_a \lor G_b = \{ G_x \mid a \lor b \leq x \}$.

In the following lemmas we investigate the conditions of a hyperlattice.

**Lemma 2.6.** $G_a \lor G_{a \land b} = G_{a \lor b} = G_{a \lor G_b} = G_{a \lor G_a} = G_{a \land G_a} = G_a \lor G_b = G_{a \land G_b} = G_{a \lor G_a} = G_{a \land b} = G_a \lor G_b = G_{a \land G_a} = G_{a \lor G_b} = G_{a \lor G_a} = G_{a \land b} = G_a \land G_b$.

**Proof.** Obvious. \( \square \)

**Lemma 2.7.** $(G_a \lor G_b) \lor G_c = G_a \lor (G_b \lor G_c)$ and $(G_a \land G_b) \land G_c = G_a \land (G_b \land G_c)$.

**Proof.** The associativity of $\land$ is obvious. We will show the associativity of $\lor$. Suppose $a, b, c \in B$. Then

$$(G_a \lor G_b) \lor G_c = \{ G_x \mid a \lor b \leq x \} \lor G_c = \bigcup_{a \lor b \leq x} G_x \lor G_c =$$

$$= \bigcup_{a \lor b \leq x} \{ G_t \mid x \lor c \leq t \} = \{ G_u \mid a \lor b \lor c \leq u \}$$

Similar argument shows that $G_a \lor (G_b \lor G_c) = \{ G_u \mid a \lor b \lor c \leq u \}$, and the result follows. \( \square \)
Lemma 2.8. \( G_a \in [G_a \lor (G_a \land G_b)] \cap [(G_a \land (G_a \lor G_b)], \) for all \( a, b \in B. \)

**Proof.** Suppose \( a, b \) are arbitrary elements of \( B, \) then we have

\[
G_a \lor (G_a \land G_b) = G_a \lor G_{a \land b} =
\]

\[
= \{G_t \mid a \lor (a \land b) \leq t\} = \{G_t \mid a \leq t\}.
\]

Therefore, \( G_a \in G_a \lor (G_a \land G_b). \) On the other hand, \( G_a = G_{a \land (a \lor b)} = G_a \land G_{a \lor b} \in G_a \land (G_a \lor G_b), \) as required. \( \square \)

**Lemma 2.9.** \( G_a \in G_a \lor G_b \) implies that \( G_b = G_a \land G_b. \)

**Proof.** Suppose \( G_a \in G_a \lor G_b, \) then there exists \( t \in B \) such that \( G_a = G_t \) and \( a \lor b \leq t. \) Thus, \( b = b \land (a \lor b) \leq b \land t \) and so \( G_b \subseteq G_{b \land t} = G_b \land G_t = G_a \land G_b. \) Therefore, \( G_b = G_a \land G_b \) and the lemma is proved. \( \square \)

We summarize the above lemmas in the following theorem:

**Theorem 2.10.** Let \( s : G \longrightarrow B \) be a function such that \( (G, \ast_s) \) constitute a hypergroup. Also, we assume that for all positive integer \( n \) and the elements \( a_1, \ldots, a_n \) of \( G, \) we have

\[
a_1 \ast_s a_2 \ast_s \ldots \ast_s a_n = \{g \in G \mid s(g) \leq s(a_1) \lor \ldots \lor s(a_n)\}.
\]

Then \( (\mathcal{L}(G), \lor, \land) \) is a hyperlattice.

We now investigate the distributivity of \( \mathcal{L}(G) \) and show this hyperlattice is not distributive, in general. In fact, we have the following example.

**Example 2.11.** There exists a function \( s : G \longrightarrow B \) such that \( (G, \ast_s) \) is a hypergroup which satisfies the conditions of Theorem 2.10, but \( \mathcal{L}(G) \) is not distributive. To see this, we assume that \( H \) is a finite group, \( \Pi_e(H) = \{o(x) \mid x \in H\} \) and \( s : P(H) \longrightarrow P(\Pi_e(H)) \) defined by \( s(A) = \{o(x) \mid x \in A\}. \) It is easy to see that the function \( s \) is onto, so by Theorem 3.6 \( \mathcal{L}(P(H)) \) is a hyperlattice. Suppose, \( H = Z_4 = \{e, a, a^2, a^3\}, \) the cyclic group of order four, and \( G = P(H). \) Then \( \Pi_e(Z_4) = \{1, 2, 4\}. \) Set \( A = \{1, 2\}, B = \{1\}, C = \{2, 4\} \) and \( D = \{2\}. \) It is clear that \( G_A \land (G_B \lor G_C) = G_A \land G_{\Pi_e(G)} = G_A \) and \( (G_A \land G_B) \lor (G_A \land G_C) = G_B \lor G_D = \{G_A, G_{\Pi_e(G)}\}. \) This shows that \( G_A \land (G_B \lor G_C) \neq (G_A \land G_B) \lor (G_A \land G_C). \) Therefore, \( \mathcal{L}(P(Z_4)) \) is a hyperlattice which is not distributive.

It is natural to ask about modularity of \( \mathcal{L}(G). \) Here, we obtain an example such that its sub-hypergroup hyperlattice is not modular.

**Example 2.12.** Assume that \( X = \{1, 2, 3, 4, 5\}, \) \( G = B = P(X) \) and \( s \) is the identity function on \( G. \) Set \( A = \{1, 2\}, B = \{1, 2, 3\} \) and \( C = \{4, 5\}. \) Then \( G_A \subseteq G_B, |G_A \lor (G_B \land G_C)| = 8 \) and \( |G_B \land (G_A \lor G_C)| = 2. \) This shows that the hyperlattice \( \mathcal{L}(G) \) is not modular.
3 About some Generalized Action

Suppose \( s : G \rightarrow B \) is a function such that \((G, \ast)\) is a hypergroup and \( A = \text{Atom}(B) \). Define the map \( \circ : G \times A \rightarrow P^*(A) \) by \( g \circ x = \{ a \in A \mid a \leq x \lor s(g) \} \). In this section, we obtain a condition on \( s \) such that \( \circ \) is a generalized action and prove that under this condition the hypergroup \((G, \ast)\) is isomorphic to a sub-hypergroup of \( M_A \).

Finally, we define a generalized action of an \( H_v \)-group on a set \( X \) as in hypergroups. We will apply the elementary properties of a generalized action and prove an inequality between the partition function \( po(n) \) and the order of the hypergroup \( M_{I(n)} \).

Lemma 3.1. Let \( B \) be a Boolean algebra and \( A = \text{Atom}(B) \). If \( s : G \rightarrow B \) is a function such that the image of \( G \) is a partition of 1, then the map \( \circ : G \times A \rightarrow P^*(A) \) defined by \( g \circ a = \{ x \in A \mid x \leq a \lor s(g) \} \) is a strong generalized action of \( G \) on \( A \).

Proof. Suppose \( g \in G \). Then it is obvious that for all \( x \in A \), we have \( x \in g \circ x \subseteq \bigcup_{a \in A} g \circ a \), i.e., \( A = \bigcup_{a \in A} g \circ a \). Thus, \( g \circ A = A \) and the condition (i) is satisfied. We now assume that \( T = \{ a \in A \mid a \leq x \lor s(g) \lor s(h) \} \) and prove that \( gh \circ x = g \circ (h \circ x) = T \). It is easy to see that \( gh \circ x \cup g \circ (h \circ x) \subseteq T \).

Suppose \( a \in T \). Then \( a \leq x \lor s(g) \lor s(h) \) and we have \( a = (a \land s(h)) \lor [a \land (x \lor s(g))] \).

We first assume that \( a \neq s(h) \), then \( a \land s(h) = 0 \) and so \( a \leq x \lor s(g) \). This shows that \( a \in g \circ x \subseteq \bigcup_{x \in gh} f \circ x = gh \circ x \). Next we assume that \( a = s(h) \). Then \( a \in h \circ x \subseteq gh \circ x \) and so \( T = gh \circ x \). Using similar argument as above, we have \( T = g \circ (h \circ x) \), proving the lemma. □

Lemma 3.2. Let \( B \) be a Boolean algebra and \( A = \text{Atom}(B) \). If \( s : G \rightarrow B \) is a one-to-one function such that the image of \( G \) is a partition of 1 and that for all \( g \in G \), there exists an atom \( x \) such that \( |g \circ x| \leq 2 \), then \((G, \ast)\) is isomorphic to a sub-hypergroup of the hypergroup \( M_A \).

Proof. By Lemma 3.1, \( \circ : G \times A \rightarrow P^*(A) \) is a strong generalized action of \( G \) on \( A \) and by Proposition 3.1 of [17] this action induced a good homomorphism \( \xi : G \rightarrow M_A \) defined by \( \xi(g)(a) = g \circ a \). It is enough to show that this homomorphism is one-to-one. To do this, suppose \( g \circ x = h \circ x \), for all \( x \in A \). By assumption, there exists an atom \( x \) such that \( |g \circ x| \geq 2 \). If \( a \neq x \) and \( a \in g \circ x \) then \( a \leq s(g) \lor x = s(h) \lor x \), and so \( a \leq x \lor (s(g) \land s(h)) \). Thus, \( a = (a \land x) \lor (a \land s(g) \land s(h)) = a \land s(g) \land s(h) \), i.e., \( a \leq s(g) \land s(h) \).

But, the image of \( G \) is a partition of 1, hence \( s(g) = s(h) \) and by injectivity of \( s \), \( g = h \). □

Suppose \( s : \Pi_d(n) \rightarrow P^*(I(n)) \) is defined by \( s(\lambda) = \text{Part}(\lambda) \). Then \((\Pi_d(n), \ast)\) is an \( H_v \)-group. Define the map \( \circ : \Pi_d(n) \times I(n) \rightarrow P^*(I(n)) \) by \( \lambda \circ k = \text{Part}(\lambda) \cup \{ k \} \). In the following simple lemma we show that the map \( \circ \) is a strong generalized action of \( \Pi_d(n) \) on the set \( I(n) \).
Lemma 3.3. The map $\circ : \Pi_d(n) \times I(n) \rightarrow P^*(I(n))$ defined by $\lambda \circ k = \text{Part}(\lambda) \cup \{k\}$ is a strong generalized action of $\Pi_d(n)$ on the set $I(n)$.

**Proof.** We first assume that $n$ is an odd positive integer, we define the partitions $\mu_i$, $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, by the following table:

<table>
<thead>
<tr>
<th>$\mu_0$</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\cdots$</th>
<th>$\mu_{\left\lfloor \frac{n}{2} \right\rfloor}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = n$</td>
<td>$n = 1 + (n - 1)$</td>
<td>$n = 2 + (n - 2)$</td>
<td>$\cdots$</td>
<td>$n = \left\lfloor \frac{n}{2} \right\rfloor + (n - \left\lfloor \frac{n}{2} \right\rfloor)$</td>
</tr>
</tbody>
</table>

Next we assume that $n$ is even, then we define the partitions $\xi_i$, $0 \leq i \leq \frac{n}{2}$, by $\xi_i = \mu_i$, for all $i < \frac{n}{2}$ and $\xi_{\frac{n}{2}}$ is the partition $n = 1 + (\frac{n}{2} - 1) + \frac{n}{2}$. Then it is clear that $\bigcup_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \text{Part}(\mu_i) = \bigcup_{i=1}^{\frac{n}{2}} \text{Part}(\xi_i) = I(n)$, and so the reproduction axiom is valid. We now assume that $\lambda, \mu$ are arbitrary partitions and $m \in I(n)$. Then we have

\[ (\lambda \ast \mu) \circ m = \bigcup_{\delta \in \lambda \ast \mu} \delta \circ m = \bigcup_{\delta \in \lambda \ast \mu} \text{Part}(\delta) \cup \{m\} = \text{Part}(\lambda) \cup \text{Part}(\mu) \cup \{m\}. \]

On the other hand,

\[ \lambda \circ (\mu \circ m) = \bigcup_{k \in \lambda \circ m} \lambda \Box k = \bigcup_{k \in \lambda \circ m} (\text{Part}(\lambda) \cup \{k\}) = \text{Part}(\lambda) \cup \text{Part}(\mu) \cup \{m\}. \]

Therefore, the map $\circ$ is a strong generalized action of the $H_0$-group $\Pi_d(n)$ on the set $I(n)$. □

Lemma 3.4. $\text{po}(n) \leq \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (2^{n-i} - 1)^n$.

**Proof.** By Euler’s partition theorem [1], $\text{po}(n) = |\Pi_d(n)|$ and by Proposition 4.1 of [17], the right hand of this inequality is the order of $M_{I(n)}$. Therefore, it is enough to show that $|\Pi_d(n)| \leq |M_{I(n)}|$. To do this, we now prove that the induced homomorphism $\eta : \Pi_d(n) \rightarrow M_{I(n)}$ by $\eta(\mu)(x) = \mu \circ x$ is one-to-one. Assume that $\eta(\mu) = \eta(\xi)$, then $\mu \circ x = \xi \circ x$, for all $x \in I(n)$. Thus, $\text{Part}(\mu) \cup \{x\} = \text{Part}(\xi) \cup \{x\}$, for all $x \in I(n)$. Choose $x \in \text{Part}(\mu)$. We have $\text{Part}(\xi) \subseteq \text{Part}(\mu) \cup \{x\} = \text{Part}(\mu)$. Similarly, $\text{Part}(\mu) \subseteq \text{Part}(\xi)$ and so $\text{Part}(\mu) = \text{Part}(\xi)$. Now since $\mu$ and $\xi$ have distinct parts, $\mu = \xi$. □
In the end of this paper, we define a generalized action of $\text{Sym}_e(G)$ on the group $G$. Suppose $\Box : \text{Sym}_e(G) \times G \rightarrow P^*(G)$ sends $(\phi, g)$ to $\phi((g))$. Then we have

$$\phi \Box (\psi \Box g) = \phi \Box \psi((g)) = \bigcup_{i \in Z} \phi \Box \psi(g^i) =$$

$$= \bigcup_{i \in Z} \phi((\psi(g^i))) = \{\phi(\psi(g^i))^j \mid i, j \in Z\}$$

and $\phi \Box g = \phi(\psi((g)) = \{\phi(\psi(g^i)) \mid i \in Z\}$. This shows that $\phi \Box g \subseteq \phi(\psi(g)).$

On the other hand, $\phi \Box G = \bigcup_{g \in G} \phi \Box g = \bigcup_{g \in G} \phi((g)) = G$, which shows that $\Box$ is a generalized action of $\text{Sym}_e(G)$ on the group $G$.

**Question 3.5.** When this generalized action is strong?

**References**


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Received January 20, 2003