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On some Hypergroups and their Hyperlattice Structures

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Abstract. Let G be a hypergroup and $\mathcal{L}(G)$ be the set of all subhypergroups of G. In this survey article, we introduce some hypergroups G from combinatorial structures and study the structure of the set $\mathcal{L}(G)$. We prove that in some cases $\mathcal{L}(G)$ has a lattice or hyperlattice structure.

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1 Introduction

First of all we will recall some algebraic definitions used in the paper. A hyperstructure is a set H together with a function $\cdot : H \times H \longrightarrow P^\star(H)$ called hyperoperation, where $P^\star(H)$ denotes the set of all non-empty subsets of H. F.Marty [18] defined a hypergroup as a hyperstructure $(H, .)$ such that the following axioms hold: (i) $(x.y).z = x.(y.z)$ for all x, y, z in H, (ii) $x.H = H.x = H$ for all x in H. The axiom (ii) is called the reproduction axiom. A commutative hypergroup (H, o) is called a join space if for all $a, b, c, d \in H$, the implication $a/b \cap c/d \neq \emptyset \Longrightarrow aod \cap boc \neq \emptyset$ is valid, in which $a/b = \{x \mid a \in xob\}$.

The concept of an H_v-group is introduced by T.Vougiouklis in [20] and it is a hyperstructure $(H, .)$ such that (i) $(x.y).z \cap x.(y.z) \neq \emptyset$, for all x, y, z in H, (ii) $x.H = H.x = H$ for all x in H. The first axiom is called weak associativity.

Following Gionfriddo [12] and Vougiouklis [20], we define a generalized permutation on a non-empty set X as a map $f : X \longrightarrow \mathcal{P}^*(X)$ such that the reproductive axiom is valid, i.e. $\cup_{x\in X} f(x) = f(X) = X$. The set of all generalized permutations on X is denoted by M_X. We now assume that $(G, \cdot)$ is a hypergroup and X is a set. The map $\odot : G \times X \longrightarrow \mathcal{P}(X)^*$ is called a generalized action of G on X if the following axioms hold:

1) For all $g, h \in G$ and $x \in X$, $(gh) \odot x \subseteq g \odot (h \odot x)$,

2) For all $g \in G$, $g \odot X = X$.

Here, for any $g \in G$ and $Y \subseteq X$, $g \odot Y$ is defined as $\cup_{x\in Y} g \odot x$, and for any $x \in X$ and $B \subseteq G$, $B \odot x$ is, by definition, equal to $\cup_{b\in B} b \odot x$. If the equality holds in the axiom 1) of definition, the generalized action is called strong (see [17]).

Following Konstantinidou and Mittas [15], we define a hyperlattice as a set H on which a hyperoperation $\vee$ and an operation $\wedge$ are defined which satisfy the following

axioms:
1. $a \in a \vee a$ and $a \wedge a = a$,
2. $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$,
3. $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,
4. $a \in [a \vee (a \wedge b)] \bigwedge [a \wedge (a \vee b)]$,
5. $a \in a \vee b$ implies that $b = a \wedge b$.

It is well known [8] that in a lattice the distributivity of the meet $(\wedge)$ with respect to the join $(\vee)$ implies the distributivity of the join with respect to the meet and vice versa, the lattice is then called distributive. But in a hyperlattice a distinction of several types of distributivity is needed. According to Konstantinidou [16], a hyperlattice $(H, \vee, \wedge)$ will be called distributive if and only if, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for all $a, b, c \in H$. Also, the hyperlattice $(H, \vee, \wedge)$ is called modular if $a \leq b$, implies that $a \vee (b \wedge c) = b \wedge (a \vee c)$, for all $c \in H$.

The second author in [2, 3] and [5], studied the construction of join spaces from some combinatorial structures. In [4], he found a new closed formula for the partition function $p(n)$. We encourage reader to consult these papers for discussion and background material.

Our notation is standard and taken mainly from [1, 8–10] and [20].

2 The Structure of some Hypergroups

Let G be a group, $Sym(G)$ be the group of all permutations on G and $Sym_e(G)$ be the stabilizer of the identity $e \in G$ in $Sym(G)$. Given two permutations ϕ and ψ from $Sym_e(G)$ and an element $g \in G$, we define a new permutation $\phi \odot_g \psi = L_{\phi(g)^{-1}} \phi L_g \psi$, where $L_{\phi(g)^{-1}}, L_g \in Sym(G)$ are left multiplications by the elements $\phi(g)^{-1}$ and g, respectively.

According to [13], a subgroup H of $Sym_e(G)$ closed under taking products of this form is called rotary closed, i.e. $H \leq Sym_e(G)$ is called rotary closed provided that $\phi \odot_g \psi \in H$, for all $\phi, \psi \in H$ and $g \in G$. A nice family of rotary closed subgroups of $Sym_e(G)$, for finite G's, comes from the theory of Cayley graphs and can be obained in the following way. Let Ω be a set of generators for a finite group G not containing the identity element e but containing x^{-1} together with every x contained in Ω. The subgroup $Rot_\Omega(G)$ of $Sym_e(G)$ of all permutations preserving e and satisfying the condition $\phi(a)^{-1}\phi(ax) \in \Omega$, for every $a \in G$ and $x \in \Omega$, is rotary closed (for details see [13] and [14]).

In this section, first we introduce a hyperoperation $\odot$ on $Sym_e(G)$ and prove that $(Sym_e(G), \odot)$ is a hypergroup. Next, we characterize the sub-hypergroups of this hypergroup. To do this, assume that $\phi, \psi \in Sym_e(G)$, we define $\phi \odot \psi = \{\phi \odot_g \psi \mid g \in G\}$.

Proposition 2.1. *$(Sym_e(G), \odot)$ is a hypergroup.*

Proof. Suppose ϕ, ψ and η are arbitrary permutations of $Sym_e(G)$. Then we have

$$(\phi \odot \psi) \odot \eta = \{\phi \odot_g \psi \mid g \in G\} \odot \eta = \bigcup_{g\in G} (\phi \odot_g \psi) \odot \eta =$$

$$= \bigcup_{g\in G} \{(\phi \odot_g \psi) \odot_h \eta \mid h \in G\} = \{(\phi \odot_g \psi) \odot_h \eta \mid g, h \in G\}.$$

Using similar argument as in above, we can show that

$$\phi \odot (\psi \odot \eta) = \{\phi \odot_g (\psi \odot_h \eta) \mid g, h \in G\}.$$

We now assume that $g, h \in G$, then we have

$$(\phi \odot_g \psi) \odot_h \eta = L_{\phi\odot_g\psi(h)^{-1}} \phi \odot_g \psi L_h \eta = L_{\phi(g\psi(h))^{-1}\phi(g)} \phi \odot_g \psi L_h \eta =$$

$$= L_{\phi(g\psi(h))^{-1}\phi(g)} L_{\phi(g)^{-1}} \phi L_g \psi L_h \eta = L_{\phi(g\psi(h))^{-1}} \phi L_g \psi L_h \eta =$$

$$= L_{\phi(g\psi(h))^{-1}} \phi L_{g\psi(h)} L_{\psi(h)^{-1}} \psi L_h \eta = \phi \odot_{g\psi(h)} (\psi \odot_h \eta) \in \phi \odot (\psi \odot \eta).$$

Therefore, $\phi \odot (\psi \odot \eta) \subseteq (\phi \odot \psi) \odot \eta$. Using similar argument we have $\phi \odot (\psi \odot \eta) \subseteq (\phi \odot \psi) \odot \eta$ and the associativity is valid. Next we assume that $\phi \in Sym_e(G)$ and we have

$$\phi \odot Sym_e(G) = \bigcup_{\psi\in Sym_e(G)} \phi \odot \psi = \bigcup_{\psi\in Sym_e(G)} \{\phi \odot_g \psi \mid g \in G\}.$$

Suppose $\delta \in Sym_e(G)$ is arbitrary and $\psi = L_{g^{-1}} \phi^{-1} L_{\phi(g)} \delta$. Then, $\phi \odot_g \psi = \delta$ and so $Sym_e(G) = \phi \odot Sym_e(G)$. Similarly, $Sym_e(G) \odot \phi = Sym_e(G)$, which completes the proof. □

In what follows, we characterize the sub-hypergroups of the hypergroup $(Sym_e(G), \odot)$.

Proposition 2.2. *Let G be a finite group and H be a non-empty subset of $Sym_e(G)$. H is a sub-hypergroup of $Sym_e(G)$ if and only if H is a rotary closed subgroup of $Sym_e(G)$.*

Proof. ($\Rightarrow$) Suppose H is a sub-hypergroup of $Sym_e(G)$. We first show that H is a closed subset of $Sym_e(G)$. To do this, suppose ϕ and ψ are elements of H. Then $\phi\psi = \phi \odot_e \psi \in \phi \odot \psi \subseteq H$ and so $\phi\psi \in H$. Next, for $\phi, \psi \in H$ and $g \in G$, $\phi \odot_g \psi \in \phi \odot \psi \subseteq H$, as desired.

($\Leftarrow$) Suppose $H \leq Sym_e(G)$ is rotary closed and $\phi \in G$. Since H is rotary closed $\phi \odot H \subseteq H$. Suppose $\psi \in H$. Put $\eta = \phi^{-1}\psi$ and $g = e$. Then, $\phi \odot_g \eta = \phi\phi^{-1}\psi = \psi \in \phi \odot \eta$ and so $H = \phi \odot H$. Similar argument shows that $H \odot \phi = H$, proving the result. □

It is a well-known fact that the set of all subgroups of a group G has a lattice structure under the ordinary operations of meet and join. In general, it is far from true that the set of all sub-hypergroups of a hypergroup has a lattice structure under these operations. In fact, the intersection of two sub-hypergroups of a hypergroup is not necessarily non-empty.

Let $\mathcal{L}(G)$ be the set of all sub-hypergroups of the hypergroup G. In what follows, we show that $Sym_e(G)$ has a lattice structure under the ordinary operations of join and meet.

Proposition 2.3. *$\mathcal{L}(Sym_e(G))$ has a lattice structure under the ordinary operations of meet and join.*

Proof. It is an easy fact that $\{1_G\}$ and $Sym_e(G)$ are rotary closed. Suppose that H and K are two rotary closed subgroups of $Sym_e(G)$. It is clear that $H \cap K$ is rotary closed. We claim that $\langle H, K\rangle$ is also rotary closed. To do this, we assume that $\psi \in H$, $\phi \in K$ and $g \in G$. Then we have:

$$\psi \odot_g \phi = L_{\psi(g)^{-1}}\psi L_g \phi = L_{\psi(g)^{-1}}\psi L_g \psi\psi^{-1}\phi = \psi \odot_g \psi\psi^{-1}\phi \in \langle H, K\rangle.$$

Also, for $\psi_1, \psi_2 \in H$, $\phi_1, \phi_2 \in K$ and $g \in G$, we have

$$\psi_1\phi_1 \odot_g \psi_2\phi_2 = L_{\psi_1\phi_1(g)^{-1}}\psi_1\phi_1 L_g \psi_2\phi_2 =$$

$$= L_{(\psi_1(\phi_1(g)))^{-1}}\psi_1 L_{\phi_1(g)}\psi_1\psi_1^{-1}L_{\phi_1(g)^{-1}}\phi_1 Lg\psi_2\phi_2 =$$
$$= (\psi_1 \odot_{\phi_1(g)} \psi_1)\psi_1^{-1}(\phi_1 \odot_g \psi_2\phi_2) \in HK \subseteq \langle H, K\rangle.$$

Using similar argument as in above, we can show that $\langle H, K\rangle$ is a rotary closed subgroup of $Sym_e(G)$. This shows that $\mathcal{L}(Sym_e(G))$ has a lattice structure under ordinary operations of join and meet. $\square$

Let G be a set, B an algebraic Boolean algebra and s a function from G into B. We define the hyperoperation $\overset{s}{\star}$ as follows:

$$a \overset{s}{\star} b = \{x \in G \mid s(x) \leq s(a) \vee s(b)\}.$$

Since for all $x, y \in G$ $\{x, y\} \subseteq x \overset{s}{\star} y$, $(G, \overset{s}{\star})$ is an H_v-group. It is also clear that the hyperoperation $\overset{s}{\star}$ is commutative.

In what follows, we study the sub-hypergroup structure of the hypergroup $(G, \overset{s}{\star})$. In some special cases we will show that the set $\mathcal{L}(G)$ has a hyperlattice structure. We also assume that $G_a = \{g \in G \mid s(g) \leq a\}$. It is easy to see that if $a \in B$ and $G_a \neq \emptyset$ then G_a is an H_v-subgroup of G. In what follows, when we write G_a, we assume that $G_a \neq \emptyset$.

Proposition 2.4. *Let B be a complete Boolean algebra and $s : G \longrightarrow B$ be a function such that $(G, \overset{s}{\star})$ constitute a hypergroup. Also, we assume that that*

$$a_1 \overset{s}{\star} a_2 \overset{s}{\star} \cdots \overset{s}{\star} a_n = \{g \in G \mid s(g) \leq s(a_1) \vee \cdots \vee s(a_n)\},$$

and H is a sub-hypergroup of G. Then there exists an element $a \in B$ such that $H = G_a$.

Proof. Let H be a sub-hypergroup of G and $a = \vee_{b \in H} s(b)$. We claim that $H = G_a$. To see this, assume $x \in H$. Then $s(x) \leq \vee_{b \in H} s(b) = a$ and so $x \in G_a$, i.e., $H \subseteq G_a$. We now assume that $x \in G_a$. Then $s(x) \leq a = \vee_{b \in H} s(b)$. Since B is algebraic, there are the elements $b_1, b_2, \cdots, b_r$ of H such that $s(x) \leq s(b_1) \vee \cdots \vee s(b_r)$. Now by assumption, $x \in \{g \in G \mid s(g) \leq s(b_1) \vee \cdots \vee s(b_r)\} = b_1 \overset{s}{\star} b_2 \overset{s}{\star} \cdots \overset{s}{\star} b_r$ and H is a sub-hypergroup of G, so $x \in H$, proving the result. □

It is clear that $G_{a \wedge b} = G_a \cap G_b$, for all $a, b \in B$. It is far from true that $G_{a \vee b} = G_a \cup G_b$. To see this, we construct an example as follows:

Example 2.5. *Suppose $G = B = P(X)$, s is the identity function, $|X| \geq 3$ and a, b, c distinct elements of X. Set $R = \{a, b\}$ and $S = \{c\}$. Then $G_R = P(R), G_S = P(S)$ and $G_{R \cup S} = P(R \cup S)$. Now it is easy to see that $G_{R \cup S} \neq G_R \cup G_S$.* □

By the results of [3] and [4], if the image of G is a $\vee$-sub-semilattice or constitutes a partition of 1, then $\mathcal{L}(G) = \{G_a \mid a \in B \& G_a \neq \emptyset\}$. In this case, we define a hyperoperation $\vee$ on $\mathcal{L}(G)$ such that $(\mathcal{L}(G), \vee, \wedge)$ constitutes a hyperlattice. To do this, we assume that $G_a \vee G_b = \{G_x \mid a \vee b \leq x\}$.

In the following lemmas we investigate the conditions of a hyperlattice.

Lemma 2.6. *$G_a \in G_a \vee G_a, G_a \wedge G_a = G_a$, $G_a \vee G_b = G_b \vee G_a$ and $G_a \wedge G_b = G_{a \wedge b} = G_b \wedge G_a$.*

Proof. Obvious. □

Lemma 2.7. $(G_a \vee G_b) \vee G_c = G_a \vee (G_b \vee G_c)$ and $(G_a \wedge G_b) \wedge G_c = G_a \wedge (G_b \wedge G_c)$.

Proof. The associativity of $\wedge$ is obvious. We will show the associativity of $\vee$. Suppose $a, b, c \in B$. Then

$$(G_a \vee G_b) \vee G_c = \{G_x \mid a \vee b \leq x\} \vee G_c = \bigcup_{a \vee b \leq x} G_x \vee G_c =$$

$$= \bigcup_{a \vee b \leq x} \{G_t \mid x \vee c \leq t\} = \{G_u \mid a \vee b \vee c \leq u\}$$

Similar argument shows that $G_a \vee (G_b \vee G_c) = \{G_u \mid a \vee b \vee c \leq u\}$, and the result follows. □

Lemma 2.8. $G_a \in [G_a \vee (G_a \wedge G_b)] \cap [(G_a \wedge (G_a \vee G_b)]$, for all $a, b \in B$.

Proof. Suppose a, b are arbitrary elements of B, then we have

$$G_a \vee (G_a \wedge G_b) = G_a \vee G_{a \wedge b} =$$

$$= \{G_t \mid a \vee (a \wedge b) \leq t\} = \{G_t \mid a \leq t\}.$$

Therefore, $G_a \in G_a \vee (G_a \wedge G_b)$. On the other hand, $G_a = G_{a \wedge (a \vee b)} = G_a \wedge G_{a \vee b} \in G_a \wedge (G_a \vee G_b)$, as required. □

Lemma 2.9. *$G_a \in G_a \vee G_b$ implies that $G_b = G_a \wedge G_b$.*

Proof. Suppose $G_a \in G_a \vee G_b$, then there exists $t \in B$ such that $G_a = G_t$ and $a \vee b \leq t$. Thus, $b = b \wedge (a \vee b) \leq b \wedge t$ and so $G_b \subseteq G_{b \wedge t} = G_b \wedge G_t = G_a \wedge G_b$. Therefore, $G_b = G_a \wedge G_b$ and the lemma is proved. □

We summarize the above lemmas in the following theorem:

Theorem 2.10. *Let $s : G \longrightarrow B$ be a function such that $(G, \overset{s}{\star})$ constitute a hypergroup. Also, we assume that for all positive integer n and the elements $a_1, \cdots, a_n$ of G, we have*

$$a_1 \overset{s}{\star} a_2 \overset{s}{\star} \cdots \overset{s}{\star} a_n = \{g \in G \mid s(g) \leq s(a_1) \vee \cdots \vee s(a_n)\}.$$

Then $(\mathcal{L}(G), \vee, \wedge)$ is a hyperlattice.

We now investigate the distributivity of $\mathcal{L}(G)$ and show this hyperlattice is not distributive, in general. In fact, we have the following example.

Example 2.11. *There exists a function $s : G \longrightarrow B$ such that $(G, \overset{s}{\star})$ is a hypergroup which satisfies the conditions of Theorem 2.10, but $\mathcal{L}(G)$ is not distributive. To see this, we assume that H is a finite group, $\Pi_e(H) = \{o(x) \mid x \in H\}$ and $s : P(H) \longrightarrow P(\Pi_e(H))$ defined by $s(A) = \{o(x) \mid x \in A\}$. It is easy to see that the function s is onto, so by Theorem 3.6 $\mathcal{L}(P(H))$ is a hyperlattice. Suppose, $H = Z_4 = \{e, a, a^2, a^3\}$, the cyclic group of order four, and $G = P(H)$. Then $\Pi_e(Z_4) = \{1, 2, 4\}$. Set $A = \{1, 2\}, B = \{1\}, C = \{2, 4\}$ and $D = \{2\}$. It is clear that $G_A \wedge (G_B \vee G_C) = G_A \wedge G_{\Pi_e(G)} = G_A$ and $(G_A \wedge G_B) \vee (G_A \wedge G_C) = G_B \vee G_D = \{G_A, G_{\Pi_e(G)}\}$. This shows that $G_A \wedge (G_B \vee G_C) \neq (G_A \wedge G_B) \vee (G_A \wedge G_C)$. Therefore, $\mathcal{L}(P(Z_4))$ is a hyperlattice which is not distributive.*

It is natural to ask about modularity of $\mathcal{L}(G)$. Here, we obtain an example such that its sub-hypergroup hyperlattice is not modular.

Example 2.12. *Assume that $X = \{1, 2, 3, 4, 5\}$, $G = B = P(X)$ and s is the identity function on G. Set $A = \{1, 2\}, B = \{1, 2, 3\}$ and $C = \{4, 5\}$. Then $G_A \subseteq G_B$, $|G_A \vee (G_B \wedge G_C)| = 8$ and $|G_B \wedge (G_A \vee G_C)| = 2$. This shows that the hyperlattice $\mathcal{L}(G)$ is not modular.*

3 About some Generalized Action

Suppose $s : G \longrightarrow B$ is a function such that $(G, \overset{s}{\star})$ is a hypergroup and $A = \mathrm{Atom}(B)$. Define the map $\odot : G \times A \longrightarrow P^\star(A)$ by $g \odot x = \{a \in A \mid a \leq x \vee s(g)\}$. In this section, we obtain a condition on s such that $\odot$ is a generalized action and prove that under this condition the hypergroup $(G, \overset{s}{\star})$ is isomorphic to a sub-hypergroup of M_A.

Finally, we define a generalized action of an H_v-group on a set X as in hypergroups. We will apply the elementary properties of a generalized action and prove an inequality between the partition function $po(n)$ and the order of the hypergroup $M_{I(n)}$.

Lemma 3.1. *Let B be a Boolean algebra and $A = Atom(B)$. If $s : G \longrightarrow B$ is a function such that the image of G is a partition of 1, then the map $\odot : G \times A \longrightarrow P^\star(A)$ defined by $g \odot a = \{x \in A \mid x \leq a \vee s(g)\}$ is a strong generalized action of G on A.*

Proof. Suppose $g \in G$. Then it is obvious that for all $x \in A$, we have $x \in g \odot x \subseteq \bigcup_{a \in A} g \Box a$, i.e., $A = \bigcup_{a \in A} g \odot a$. Thus, $g \odot A = A$ and the condition (i) is satisfied. We now assume that $T = \{a \in A \mid a \leq x \vee s(g) \vee s(h)\}$ and prove that $gh \odot x = g \odot (h \odot x) = T$. It is easy to see that $gh \odot x \cup g \odot (h \odot x) \subseteq T$. Suppose $a \in T$. Then $a \leq x \vee s(g) \vee s(h)$ and we have $a = (a \wedge s(h)) \vee [a \wedge (x \vee s(g)]$. We first assume that $a \neq s(h)$, then $a \wedge s(h) = 0$ and so $a \leq x \vee s(g)$. This shows that $a \in g \Box x \subseteq \bigcup_{t \in gh} t \Box x = gh \Box x$. Next we assume that $a = s(h)$. Then $a \in h \odot x \subseteq gh \odot x$ and so $T = gh \odot x$. Using similar argument as above, we have $T = g \odot (h \odot x)$, proving the lemma. □

Lemma 3.2. *Let B be a Boolean algebra and $A = Atom(B)$. If $s : G \longrightarrow B$ is a one-to-one function such that the image of G is a partition of 1 and that for all $g \in G$, there exists an atom x such that $|g \odot x| \leq 2$, then $(G, \overset{s}{\star})$ is isomorphic to a sub-hypergroup of the hypergroup M_A.*

Proof. By Lemma 3.1, $\odot : G \times A \longrightarrow P^\star(A)$ is a strong generalized action of G on A and by Proposition 3.1 of [17] this action induced a good homomorphism $\xi : G \longrightarrow M_A$ defined by $\xi(g)(a) = g \odot a$. It is enough to show that this homomorphism is one-to-one. To do this, suppose $g \odot x = h \odot x$, for all $x \in A$. By assumption, there exists an atom x such that $|g \odot x| \geq 2$. If $a \neq x$ and $a \in g \odot x$ then $a \leq s(g) \vee x = s(h) \vee x$, and so $a \leq x \vee (s(g) \wedge s(h))$. Thus, $a = (a \wedge x) \vee (a \wedge s(g) \wedge s(h)) = a \wedge s(g) \wedge s(h)$, i.e., $a \leq s(g) \wedge s(h)$. But, the image of G is a partition of 1, hence $s(g) = s(h)$ and by injectivity of s, $g = h$. □

Suppose $s : \Pi_d(n) \longrightarrow P^\star(I(n))$ is defined by $s(\lambda) = Part(\lambda)$. Then $(\Pi_d(n), \overset{s}{\star})$ is an H_v-group. Define the map $\odot : \Pi_d(n) \times I(n) \longrightarrow P^\star(I(n))$ by $\lambda \odot k = Part(\lambda) \cup \{k\}$. In the following simple lemma we show that the map $\odot$ is a strong generalized action of $\Pi_d(n)$ on the set $I(n)$.

Lemma 3.3. *The map* $\odot : \Pi_d(n) \times I(n) \longrightarrow P^\star(I(n))$ *defined by* $\lambda \odot k = Part(\lambda) \cup \{k\}$ *is a strong generalized action of* $\Pi_d(n)$ *on the set* $I(n)$.

Proof. We first assume that n is an odd positive integer, we define the partitions μ_i, $0 \leq i \leq [\frac{n}{2}]$, by the following table:

μ_0	μ_1	μ_2	$\cdots$	$m_{[\frac{n}{2}]}$
$n = n$	$n = 1 + (n-1)$	$n = 2 + (n-2)$	$\cdots$	$n = [\frac{n}{2}] + (n - [\frac{n}{2}])$

Next we assume that n is even, then we define the partitions ξ_i, $0 \leq i \leq \frac{n}{2}$, by $\xi_i = \mu_i$, for all $i < \frac{n}{2}$ and $\xi_{\frac{n}{2}}$ is the partition $n = 1 + (\frac{n}{2} - 1) + \frac{n}{2}$. Then it is clear that $\bigcup_{i=1}^{[\frac{n}{2}]} Part(\mu_i) = \bigcup_{i=1}^{[\frac{n}{2}]} Part(\xi_i) = I(n)$, and so the reproduction axiom is valid. We now assume that λ, μ are arbitrary partitions and $m \in I(n)$. Then we have

$$(\lambda \overset{s}{\star} \mu) \odot m = \bigcup_{\delta \in \lambda \overset{s}{\star} \mu} \delta \odot m =$$

$$= \bigcup_{\delta \in \lambda \overset{s}{\star} \mu} Part(\delta) \cup \{m\} = Part(\lambda) \cup Part(\mu) \cup \{m\}.$$

On the other hand,

$$\lambda \odot (\mu \odot m) = \bigcup_{k \in \lambda \odot m} \lambda \Box k =$$

$$= \bigcup_{k \in \lambda \odot m} (Part(\lambda) \cup \{k\}) = Part(\lambda) \cup Part(\mu) \cup \{m\}.$$

Therefore, the map $\odot$ is a strong generalized action of the H_v-group $\Pi_d(n)$ on the set $I(n)$. $\Box$

Lemma 3.4. $po(n) \leq \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (2^{n-i} - 1)^n$.

Proof. By Euler's partition theorem [1], $po(n) = |\Pi_d(n)|$ and by Proposition 4.1 of [17], the right hand of this inequality is the order of $M_{I(n)}$. Therefore, it is enough to show that $|\Pi_d(n)| \leq |M_{I(n)}|$. To do this, we now prove that the induced homomorphism $\eta : \Pi_d(n) \longrightarrow M_{I(n)}$ by $\eta(\mu)(x) = \mu \odot x$ is one-to-one. Assume that $\eta(\mu) = \eta(\xi)$, then $\mu \odot x = \xi \odot x$, for all $x \in I(n)$. Thus, $Part(\mu) \cup \{x\} = Part(\xi) \cup \{x\}$, for all $x \in I(n)$. Choose $x \in Part(\mu)$. We have $Part(\xi) \subseteq Part(\xi) \cup \{x\} = Part(\mu)$. Similarly, $Part(\mu) \subseteq Part(\xi)$ and so $Part(\mu) = Part(\xi)$. Now since μ and ξ have distinct parts, $\mu = \xi$. $\Box$

In the end of this paper, we define a generalized action of $Sym_e(G)$ on the group G. Suppose $\Box : Sym_e(G) \times G \longrightarrow P^{\star}(G)$ sends (ϕ, g) to $\phi(\langle g\rangle)$. Then we have

$$\phi\Box(\psi\Box g) = \phi\Box\psi(\langle g\rangle) = \bigcup_{i\in Z} \phi\Box\psi(g^i) =$$

$$= \bigcup_{i\in Z} \phi(\langle\psi(g^i)\rangle) = \{\phi(\psi(g^i))^j \mid i, j \in Z\}$$

and $\phi\psi\Box g = \phi\psi(\langle g\rangle) = \{\phi(\psi(g^i)) \mid i \in Z\}$. This shows that $\phi\psi\Box g \subseteq \phi\Box(\psi\Box g)$. On the other hand, $\phi\Box G = \bigcup_{g\in G} \phi\Box g = \bigcup_{g\in G} \phi(\langle g\rangle) = G$, which shows that $\Box$ is a generalized action of $Sym_e(G)$ on the group G.

Question 3.5. When this generalized action is strong?

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