BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(43), 2003, Pages 104–106
ISSN 1024–7696

On regularization of singular operators with Carleman shift

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Abstract. The paper is devoted to regularization of some singular integral operators. The necessary and sufficient conditions by which the singular operators with Carleman shift and conjugation admits the equivalent regularization are found.

Mathematics subject classification: 45E05.
Keywords and phrases: Regularization, noetherian singular operators, singular operators with conjugation, singular operators with shift.

Let $A \in L(\mathfrak{L})$, where $\mathfrak{L}$ is the Banach space. An operator $M \in L(\mathfrak{L})$ is said to be regularizing for A in a space $\mathfrak{L}$, if the operators $MA - I$ and $AM - I$ are compact in $\mathfrak{L}$. The class of linear bounded operators admitting regularization is attracted by that, for the operators of this class and only for them hold the following properties (F.Neother theorems):

1) the equations $Ax = 0$ and $A^*\varphi = 0$ have a finite number of linear independent solutions;

2) the equations $Ax = y$ is solvable if and only if its right –hand side is ortogonal to each of solutions of the equation $A^*\varphi = 0$;

The operator $A \in L(\mathfrak{L})$ satisfying conditions 1) and 2) is called the Fredholm operator and the number

$$Ind\, A = dimker\, A - dimker\, A^* \tag{1}$$

is called its index.

If it is known the regularizing operator M for A, then the solution of the equation

$$Ax = y \tag{2}$$

can be reduced to solving the equation

$$MAx = My, \tag{3}$$

in which the operator $MA - I$ is compact.

For investigation of equation (3) can be applied many methods developed for inversion of operators $I + T$, where T is compact.

Of course a special interest represent the case when the equations (2) and (3) are equivalent by any vector y. This means that equations (2) and (3) are solvable simultaneously and have the same solutions. This happens to be if and only if $Ker\, M = \{0\}$.

We say that the operator A admits an equivalent regularization if it has regularizing operator M for which the equations (2) and (3) are equivalent for all $y \in \mathfrak{L}$. In this case the operator M is called equivalent regularizing operator for A.

From what has been said above it follows that the operator M is equivalent regularizing operator for A, if it is regularizing for A and reversible from the left.

Theorem 1 (see [1]). *The operator $A \in L(\mathfrak{L})$ admits an equivalent regularizing if and only if*

$$IndA \geq 0. \tag{4}$$

Next we shall consider singular integral operators with shift

$$A = aP + bQ + (cP + dQ)V, \tag{5}$$

where $a, b, c, d \in C(\Gamma)$, Γ is a closed Liapunov contour, P and Q are Riesz operators and V is the operator of Carleman shift: $(V\varphi)(t) = \varphi(\alpha(t))$, $\alpha' \in H_\mu(\Gamma)$ and $\alpha(\alpha(t)) \equiv t$.

Theorem 2. *Let α preserve the orientation on Γ*

$$\Delta_1 = c(t)\tilde{c}(t) + a(t)\tilde{a}(t), \quad \Delta_2 = d(t)\tilde{d}(t) + b(t)\tilde{b}(t),$$

where $\tilde{f}(t) = f(\alpha(t))$ $(\alpha(t) \not\equiv t)$. The operator A assume the regularization in a space $L_p(\Gamma)$ if and only if

$$\Delta_1(t) \neq 0, \quad \Delta_2(t) \neq 0 \quad \forall t \in \Gamma. \tag{6}$$

Theorem 3. *The operator A admits an equivalent regularizing in $L_p(\Gamma)$ if and only if the conditions (6) hold and*

$$ind\frac{\Delta_1(t)}{\Delta_2(t)} \leq 0. \tag{7}$$

Theorem 4. *Let α change the orientation on Γ. The operator A admits regularization in $L_p(\Gamma)$ if and only if*

$$\Delta(t) = a(t)\tilde{b}(t) - c(t)\tilde{d}(t) \neq 0 \quad \forall t \in \Gamma. \tag{8}$$

Theorem 5. *The operator A admits equivalent regularization if and only if the conditions (8) are satisfied and*

$$ind\Delta(t) \leq 0. \tag{9}$$

Let us consider the singular operator of the following form

$$B = aP + bQ + (cP + dQ)W \tag{10}$$

with complex conjugation $W\varphi(t) = \overline{\varphi}(t)$. For the operator B are valid theorems 4 and 5 in which a function $\tilde{f}(t)$ is replaced by $\overline{f}(t)$.

By proving theorems 2 and 4 the theory of normalized rings is applied (see [2]) and local principle [3]. At the end we mention that enough part of theorems 2 and 3 are contained in monograph [4].

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Received December 2, 2003