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On rational bases of $GL(2,\mathbb{R})$-comitants of planar polynomial systems of differential equations*

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Abstract. The linear transformations of autonomous planar polynomial systems of differential equations which reduce these systems to the canonical forms with coefficients expressed as rational functions of $GL(2,\mathbb{R})$-comitants and $GL(2,\mathbb{R})$-invariants are established. Such canonical forms for general quadratic and cubic systems are constructed in concrete forms. Using constructed canonical forms for polynomial systems some rational bases of $GL(2,\mathbb{R})$-comitants depending on the coordinates of one vector are obtained.

Mathematics subject classification: 34C14, 34C05, 58F14.
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1 Preliminary information and notations

Let us consider the system of differential equations

$$\frac{dx^j}{dt} = \sum_{m=0}^{n} P_m^j(x^1, x^2) \quad (j = 1, 2), \tag{1}$$

where P_m^j are homogeneous polynomials in x^1 and x^2 with real coefficients. System (1) can be written in the following form:

$$\frac{dx^j}{dt} = \sum_{m=0}^{n}\sum_{i=0}^{m} \binom{m}{i} a^j_{m-i,i}(x^1)^{m-i}(x^2)^i \quad (j = 1, 2). \tag{2}$$

Let $GL(2,\mathbb{R})$ be the group of linear homogeneous non-degenerate (centroaffine [2]) transformations

$$y^r = q^r_l x^l, \quad \Delta_q = \det(q^r_l) \neq 0 \quad (r, l = 1, 2) \tag{3}$$

of the phase plane $\mathbb{R}^2$ of system (1), where $y = \binom{y^1}{y^2}$ is a new vector and $q = \begin{pmatrix} q^1_1 & q^1_2 \\ q^2_1 & q^2_2 \end{pmatrix}$. Let us denote $p = q^{-1}$, $p = \begin{pmatrix} p^1_1 & p^1_2 \\ p^2_1 & p^2_2 \end{pmatrix}$. Applying the transformation (3) system (1) well be brought to the system

$$\frac{dy^j}{dt} = \sum_{m=0}^{n}\sum_{i=0}^{m} \binom{m}{i} b^j_{m-i,i}(y^1)^{m-i}(y^2)^i \quad (j = 1, 2). \tag{4}$$

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Let $p_1,\ p_2,\ \alpha_1$ and α_2 be, respectively, the vectors

$$p_1 = \begin{pmatrix} p_1^1 \\ p_1^2 \end{pmatrix}, \quad p_2 = \begin{pmatrix} p_2^1 \\ p_2^2 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} p_1^1 y^1 \\ p_1^2 y^1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} p_2^1 y^2 \\ p_2^2 y^2 \end{pmatrix}$$

and denote by A the $(n+1)(n+2)$-dimensional coefficient space of system (1).

Definition 1. [1] *A polynomial $K(a,u,v)$ in coefficients of system (1) and coordinates of vectors $u,v \in \mathbb{R}^2$ is called a comitant of system (1) with respect to a group Q if there exists a function $\lambda : Q \to \mathbb{R}$ such that*

$$K(r_q(a), q\cdot u, q\cdot v) \equiv \lambda(q) K(a,u,v)$$

for every $q \in Q,\ a \in A$ and $u,v \in \mathbb{R}^2$.

If Q is the group $GL(2,\mathbb{R})$, then the comitant is called $GL(2,\mathbb{R})$-comitant or centroaffine comitant. In what follows only GL-comitants are considered.

The function $\lambda(q)$ is called a multiplicator. It is known [2] that the function $\lambda(q)$ has the form $\lambda(q) = \Delta_q^{-g}$, where g is an integer, which is called the weight of the comitant $K(a,u,v)$. If $g = 0$, then the comitant is called absolute, otherwise it is relative.

If a comitant does not depend on the coordinates of the vectors u and v, then it is called *invariant*.

We say that a comitant $K(a,u,v)$ has the *character* $(r_1,\ r_2;\ g;\ d)$ if it has the weight g, the degree d with respect to the coefficients of system (1) and the degree r_1 (respectively, r_2) with respect to the coordinates of the vector $u \in \mathbb{R}^2$ (respectively, $v \in \mathbb{R}^2$). In the case when a comitant depends on the coordinates of one vector we will denote its character by the triple $(r;\ g;\ d)$.

Definition 2. *The set S of comitants is called a rational on $M \subseteq A$ basis of comitants for system (1) with respect to a group Q if any comitant of system (1) with respect to the group Q can be expressed as a rational function of elements of the set S.*

Definition 3. *A rational basis on $M \subseteq A$ of comitants for system (1) with respect to a group Q is called minimal if by the removal from it of any comitant it ceases to be a rational basis.*

Let f and φ be polynomials in the coordinates of the vector $u = (u^1, u^2) \in \mathbb{R}^2$ of the degrees r and ρ, respectively.

Definition 4. *The polynomial*

$$(f,\varphi)^{(k)} = \frac{(r-k)!(\rho-k)!}{r!\rho!} \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial (u^1)^{k-h} \partial (u^2)^h} \frac{\partial^k \varphi}{\partial (u^1)^h \partial (u^2)^{k-h}}$$

is called the transvectant of index k of polynomials f and φ.

Property 1. [3] *If polynomials f and φ are GL-comitants of system (1) with the characters $(r;\ g_f;\ d_f)$ and $(\rho;\ g_\varphi;\ d_\varphi)$, respectively, then the transvectant of the index $k \le \min(r,\rho)$ is a GL-comitant of the system (1) with the character $(r+\rho-2k;\ g_f+g_\varphi+k;\ d_f+d_\varphi)$.*

Let $K(a,u,v)$ be a GL-comitant of system (1) of the character $(r_1,\ r_2;\ g;\ d)$. We denote by $D^k_{u,v}$ the polarizing operator [4, 5]:

$$D^k_{u,v}K(a,u,v) = \frac{(r_1-k)!}{r_1!}\left(v^1\frac{\partial}{\partial u^1}+v^2\frac{\partial}{\partial u^2}\right)^k K(a,u,v)$$
$$= \frac{(r_1-k)!}{r_1!}\sum_{h=0}^{k}\binom{k}{h}\frac{\partial^k K(a,u,v)}{\partial(u^1)^{k-h}\partial(u^2)^h}(v^1)^{k-h}(v^2)^h \quad (k\le r_1).$$

The following properties will be useful in what follows.

Property 2. *If $K(a,u,v)$ is a GL-comitant of system (1) of the character $(r_1,\ r_2;\ g;\ d)$, then $D^k_{u,v}K(a,u,v)$ is also a GL-comitant of system (1) with the character $(r_1-k,\ r_2+k;\ g;\ d)$.*

Property 3. *If $P(x^1,x^2)$ is a homogeneous polynomial of the degree m with respect to the coordinates of the vector $x\in\mathbb{R}^2$ and $\alpha=\alpha_1+\alpha_2$, $\alpha,\ \alpha_1,\ \alpha_2\in\mathbb{R}^2$, then*

$$D^m_{x,\alpha}P(x^1,x^2) = D^m_{x,\alpha_1+\alpha_2}P(x^1,x^2) = \sum_{i=0}^{m}\binom{m}{i}D^{m-i}_{x,\alpha_1}D^i_{x,\alpha_2}P(x^1,x^2).$$

We shall consider the following polynomials:

$$R_i = P_i^1x^2 - P_i^2x^1,\quad i=0,...,n;\qquad S_i=\frac{1}{i}\left(\frac{\partial P_i^1}{\partial x^1}+\frac{\partial P_i^2}{\partial x^2}\right),\quad i=1,...,n, \tag{5}$$

which in fact are GL-comitants of the first degree with respect to the coefficients of system (1). By using the comitants (5) system (1) can be written in the form

$$\begin{aligned}\frac{dx^1}{dt} &= \sum_{m=0}^{n}\frac{1}{m+1}\left(\frac{\partial R_m(x^1,x^2)}{\partial x^2}+mx^1S_m(x^1,x^2)\right),\\ \frac{dx^2}{dt} &= \sum_{m=0}^{n}\frac{1}{m+1}\left(-\frac{\partial R_m(x^1,x^2)}{\partial x^1}+mx^2S_m(x^1,x^2)\right).\end{aligned} \tag{6}$$

The following GL-comitants of the general cubic system of differential equations (i.e., system (1) for $n=3$) constructed by using comitants (5) and the notion of transvectant are considered:

$$
\begin{array}{llll}
K_1 = R_0, & (1;-1;\ 1); & K_2 = R_1, & (2;-1;\ 1);\\
K_3 = S_1, & (0;\ \ 0;\ 1); & K_4 = R_2, & (3;-1;\ 1);\\
K_5 = S_2, & (1;\ \ 0;\ 1); & K_6 = R_3, & (4;-1;\ 1);\\
K_7 = S_3, & (2;\ \ 0;\ 1); & K_8 = 2(R_3,R_3)^{(4)}, & (0;\ \ 2;\ 2);\\
K_9 = 8(R_3,R_3)^{(2)}, & (4;\ \ 0;\ 2); & K_{10} = 4(R_3,R_2)^{(3)}, & (1;\ \ 1;\ 2);\\
K_{11} = 6(R_3,R_2)^{(2)}, & (3;\ \ 0;\ 2); & K_{12} = 12(R_3,R_2)^{(1)}, & (5;-1;\ 2);\\
K_{13} = 4(R_3,R_1)^{(2)}, & (2;\ \ 0;\ 2); & K_{14} = 4(R_3,R_1)^{(1)}, & (4;-1;\ 2);\\
K_{15} = (R_3,S_3)^{(2)}, & (2;\ \ 1;\ 2); & K_{16} = 4(R_3,S_3)^{(1)}, & (4;\ \ 0;\ 2);\\
K_{17} = \frac{9}{2}(R_2,R_2)^{(2)}, & (2;\ \ 0;\ 2); & K_{18} = 3(R_2,R_1)^{(2)}, & (1;\ \ 0;\ 2);\\
K_{19} = 6(R_2,R_1)^{(1)}, & (3;-1;\ 2); & K_{20} = 3(R_2,S_3)^{(2)}, & (1;\ \ 1;\ 2);\\
K_{21} = 3(R_2,S_3)^{(1)}, & (3;\ \ 0;\ 2); & K_{22} = (R_1,S_3)^{(2)}, & (0;\ \ 1;\ 2);\\
K_{23} = 2(R_1,S_3)^{(1)}, & (2;\ \ 0;\ 2); & K_{24} = \frac{1}{2}(S_3,S_3)^{(2)}, & (0;\ \ 2;\ 2);\\
K_{25} = (R_0,S_2)^{(1)}, & (0;\ \ 0;\ 2); & K_{26} = 2(R_1,S_2)^{(1)}, & (1;\ \ 0;\ 2);\\
K_{27} = 3(R_2,R_0)^{(1)}, & (2;-1;\ 2); & K_{28} = 3(R_2,S_2)^{(1)}, & (2;\ \ 0;\ 2);\\
K_{29} = 4(R_3,R_0)^{(1)}, & (3;-1;\ 2); & K_{30} = 4(R_3,S_2)^{(1)}, & (3;\ \ 0;\ 2);\\
K_{31} = (S_3,R_0)^{(1)}, & (1;\ \ 0;\ 2); & K_{32} = (S_3,S_2)^{(1)}, & (1;\ \ 1;\ 2);\\
K_{33} = 32((R_3,R_3)^{(2)},R_3)^{(1)}, & (6;\ \ 0;\ 3); & K_{34} = ((R_3,S_3)^{(2)},S_3)^{(2)}, & (0;\ \ 3;\ 3);\\
K_{35} = ((R_3,S_3)^{(2)},S_3)^{(1)}, & (2;\ \ 2;\ 3); & K_{36} = 27((R_2,R_2)^{(2)},R_2)^{(1)}, & (3;\ \ 0;\ 3);\\
K_{37} = 3((R_2,S_3)^{(2)},S_3)^{(1)}, & (1;\ \ 2;\ 3); & K_{38} = 3((R_2,R_2)^{(2)},S_2)^{(1)}, & (1;\ \ 1;\ 3);\\
K_{39} = 3((R_2,R_1)^{(2)},S_2)^{(1)}, & (0;\ \ 1;\ 3); & K_{40} = 18((R_2,R_1)^{(1)},S_2)^{(1)}, & (2;\ \ 0;\ 3);\\
K_{41} = ((R_1,R_0)^{(1)},R_0)^{(1)}, & (0;-1;\ 3); & K_{42} = 2((R_1,R_0)^{(1)},S_2)^{(1)}, & (0;\ \ 0;\ 3);\\
K_{43} = ((R_1,S_2)^{(1)},S_2)^{(1)}, & (0;\ \ 1;\ 3); & K_{44} = ((R_2,S_2)^{(1)},S_2)^{(1)}, & (1;\ \ 1;\ 3);\\
K_{45} = ((S_3,R_0)^{(1)},R_0)^{(1)}, & (0;\ \ 0;\ 3); & K_{46} = ((S_3,R_0)^{(1)},S_2)^{(1)}, & (0;\ \ 1;\ 3);
\end{array}
$$

$$
\begin{array}{ll}
K_{47} = ((S_3,S_2)^{(1)},S_2)^{(1)}, & (0;\ \ 2;\ 3);\\
K_{48} = \frac{3}{2}(((R_2,R_2)^{(2)},S_2)^{(1)},S_2)^{(1)}, & (0;\ \ 2;\ 4);\\
K_{49} = (((R_2,R_0)^{(1)},R_0)^{(1)},R_0)^{(1)}, & (0;-1;\ 4);\\
K_{50} = 3(((R_2,R_0)^{(1)},R_0)^{(1)},S_2)^{(1)}, & (0;\ \ 0;\ 4);\\
K_{51} = (((R_2,R_0)^{(1)},S_2)^{(1)},S_2)^{(1)}, & (0;\ \ 1;\ 4);\\
K_{52} = (((R_2,S_2)^{(1)},S_2)^{(1)},S_2)^{(1)}, & (0;\ \ 2;\ 4);\\
K_{53} = 2((((R_2,R_1)^{(1)},S_2)^{(1)},S_2)^{(1)},S_2)^{(1)}, & (0;\ \ 2;\ 5);\\
K_{54} = ((((R_3,R_0)^{(1)},R_0)^{(1)},R_0)^{(1)},R_0)^{(1)}, & (0;-1;\ 5);\\
K_{55} = 4((((R_3,R_0)^{(1)},R_0)^{(1)},R_0)^{(1)},S_2)^{(1)}, & (0;\ \ 0;\ 5);\\
K_{56} = 2((((R_3,R_0)^{(1)},R_0)^{(1)},S_2)^{(1)},S_2)^{(1)}, & (0;\ \ 1;\ 5);\\
K_{57} = 4((((R_3,R_0)^{(1)},S_2)^{(1)},S_2)^{(1)},S_2)^{(1)}, & (0;\ \ 2;\ 5);\\
K_{58} = ((((R_3,S_2)^{(1)},S_2)^{(1)},S_2)^{(1)},S_2)^{(1)}, & (0;\ \ 3;\ 5);\\
K_{59} = (((((R_2,R_2)^{(2)},R_2)^{(1)},S_2)^{(1)},S_2)^{(1)},S_2)^{(1)}, & (0;\ \ 3;\ 6).
\end{array}
$$

Here besides the expression for each constructed GL-comitant its corresponding character is indicated.

2 The transformations by using two $GL(2,\mathbb{R})$-comitants

Theorem 1. *Let $U(a,w)$ and $V(a,w)$ be GL-comitants of the system (1) with the characters $(r_U; g_U; d_U)$ and $(r_V; g_V; d_V)$, respectively, and assume that the relation $(V,U)^{(1)} \not\equiv 0$ holds. Applying the transformation*

$$
\begin{aligned}
y^1 &= \frac{1}{r_V r_U (V,U)^{(1)}} \, \frac{\partial V(a,w)}{\partial w^1} \, x^1 + \frac{1}{r_V r_U (V,U)^{(1)}} \, \frac{\partial V(a,w)}{\partial w^2} \, x^2, \\
y^2 &= \frac{\partial U(a,w)}{\partial w^1} \, x^1 + \frac{\partial U(a,w)}{\partial w^2} \, x^2
\end{aligned} \tag{7}
$$

system (1) can be brought to the system (4) with the coefficients

$$
b^j_{m-i,i} = \frac{T^j_{m-i,i}(a,w)}{\left(r_V r_U (V,U)^{(1)}\right)^{i-j+2}} \quad (j=1,2;\ m=0,\cdots,n;\ i=0,\cdots,m), \tag{8}
$$

where $T^j_{m-i,i}(a,w)$ is a GL-comitant of system (1) with the character

$$
\begin{gathered}
((m-i+j-1)(r_U-1)+(i-j+2)(r_V-1); \\
-1+(m-i+j-1)(g_U+1)+(i-j+2)(g_V+1); \\
1+(m-i+j-1)d_U+(i-j+2)d_V).
\end{gathered} \tag{9}
$$

Moreover,

$$
T^j_{m-i,i}(a,w) = (-1)^{3-j}\left(p^2_{3-j} D^{m-i}_{p_2,p_1} P^1_m(p^1_2,p^2_2) - p^1_{3-j} D^{m-i}_{p_2,p_1} P^2_m(p^1_2,p^2_2)\right)\Bigg|_{\substack{p_1 = \left(\frac{\partial U(a,w)}{\partial w^2},\ -\frac{\partial U(a,w)}{\partial w^1}\right) \\ p_2 = \left(-\frac{\partial V(a,w)}{\partial w^2},\ \frac{\partial V(a,w)}{\partial w^1}\right)}}.
$$

Firstly, we shall prove two lemmas.

Lemma 1. *Applying the transformation (3) to system (1), we obtain system (4) with the coefficients*

$$
b^j_{m-i,i} = \frac{L^j_{m-i,i}(a,p_1,p_2)}{\Delta_p} \quad (j=1,2;\ m=0,\cdots,n;\ i=0,\cdots,m), \tag{10}
$$

where

$$
\begin{aligned}
L^j_{m-i,i}(a,p_1,p_2) &= (-1)^{3-j}\left(p^2_{3-j} D^{m-i}_{p_2,p_1} P^1_m(p^1_2,p^2_2) - p^1_{3-j} D^{m-i}_{p_2,p_1} P^2_m(p^1_2,p^2_2)\right) \\
&= (-1)^{3-j}\left(p^2_{3-j} D^{i}_{p_1,p_2} P^1_m(p^1_1,p^2_1) - p^1_{3-j} D^{i}_{p_1,p_2} P^2_m(p^1_1,p^2_1)\right)
\end{aligned} \tag{11}
$$

is a GL-comitant of system (1) with the character $(m-i+j-1,\ i-j+2;\ -1;\ 1)$.

Proof. Performing transformation (3) for system (1) and taking into consideration the property 3, we have

$$\frac{dy^j}{dt} = q_1^j \sum_{m=0}^{n} P_m^1(x^1,x^2) + q_2^j \sum_{m=0}^{n} P_m^2(x^1,x^2)\Bigg|_{\substack{x^1 = p_1^1 y^1 + p_2^1 y^2 \\ x^2 = p_1^2 y^1 + p_2^2 y^2}}$$
$$= q_1^j \sum_{m=0}^{n} D^m_{x,\alpha_1+\alpha_2} P_m^1(x^1,x^2) + q_2^j \sum_{m=0}^{n} D^m_{x,\alpha_1+\alpha_2} P_m^2(x^1,x^2)$$
$$= q_1^j \sum_{m=0}^{n}\sum_{i=0}^{m} \binom{m}{i} D^{m-i}_{x,\alpha_1} D^i_{x,\alpha_2} P_m^1(x^1,x^2) + q_2^j \sum_{m=0}^{n}\sum_{i=0}^{m} \binom{m}{i} D^{m-i}_{x,\alpha_1} D^i_{x,\alpha_2} P_m^2(x^1,x^2)$$
$$= q_1^j \sum_{m=0}^{n}\sum_{i=0}^{m} \binom{m}{i} D^{m-i}_{x,p_1} D^i_{x,p_2} P_m^1(x^1,x^2)(y^1)^{m-i}(y^2)^i +$$
$$q_2^j \sum_{m=0}^{n}\sum_{i=0}^{m} \binom{m}{i} D^{m-i}_{x,p_1} D^i_{x,p_2} P_m^2(x^1,x^2)(y^1)^{m-i}(y^2)^i$$
$$= \sum_{m=0}^{n}\sum_{i=0}^{m} \binom{m}{i} \left(q_1^j D^{m-i}_{x,p_1} D^i_{x,p_2} P_m^1(x^1,x^2) + q_2^j D^{m-i}_{x,p_1} D^i_{x,p_2} P_m^2(x^1,x^2) \right) (y^1)^{m-i}(y^2)^i.$$

So, system (4) has the coefficients:

$$\begin{aligned} b^j_{m-i,i} &= q_1^j D^{m-i}_{x,p_1} D^i_{x,p_2} P_m^1(x^1,x^2) + q_2^j D^{m-i}_{x,p_1} D^i_{x,p_2} P_m^2(x^1,x^2) \\ &= q_1^j D^{m-i}_{p_2,p_1} P_m^1(p_2^1,p_2^2) + q_2^j D^{m-i}_{p_2,p_1} P_m^2(p_2^1,p_2^2) \\ &= q_1^j D^i_{p_1,p_2} P_m^1(p_1^1,p_1^2) + q_2^j D^i_{p_1,p_2} P_m^2(p_1^1,p_1^2). \end{aligned} \tag{12}$$

Taking into account that $q = p^{-1}$ we have

$$q = \frac{1}{\Delta_p} \begin{pmatrix} p_2^2 & -p_2^1 \\ -p_1^2 & p_1^1 \end{pmatrix}, \tag{13}$$

where $\Delta_p = \det(p_l^r)$ $(r,l = 1,2)$. By (12) and (13), we have

$$\begin{aligned} b^j_{m-i,i} &= (-1)^{3-j} \frac{1}{\Delta_p} \left(p^2_{3-j} D^{m-i}_{p_2,p_1} P_m^1(p_2^1,p_2^2) - p^1_{3-j} D^{m-i}_{p_2,p_1} P_m^2(p_2^1,p_2^2) \right) \\ &= (-1)^{3-j} \frac{1}{\Delta_p} \left(p^2_{3-j} D^i_{p_1,p_2} P_m^1(p_1^1,p_1^2) - p^1_{3-j} D^i_{p_1,p_2} P_m^2(p_1^1,p_1^2) \right). \end{aligned} \tag{14}$$

Thus, the form (10), (11) of the coeffitients is established.

Next we shall prove that the polynomials $L^j_{m-i,i}(a,p_1,p_2)$ (see (10)) are GL-comitants of the indicated characters. From (14) and (6) for $b^1_{m-i,i}$ and $b^2_{m-i,i}$ we obtain, respectively,

$$\begin{aligned} b^1_{m-i,i} &= \frac{1}{\Delta_p} D^i_{p_1,p_2}\left(p_2^2 P^1_m\left(p_1^1,p_1^2\right) - p_2^1 P^2_m\left(p_1^1,p_1^2\right)\right) \\ &= \frac{1}{\Delta_p} D^i_{p_1,p_2}\left(\frac{1}{m+1}p_2^2\left(\frac{\partial R_m(p_1^1,p_1^2)}{\partial p_1^2} + m p_1^1 S_m(p_1^1,p_1^2)\right) - \right. \\ &\qquad \left. \frac{1}{m+1}p_2^1\left(-\frac{\partial R_m(p_1^1,p_1^2)}{\partial p_1^1} + m p_1^2 S_m(p_1^1,p_1^2)\right)\right) \\ &= \frac{1}{\Delta_p} D^i_{p_1,p_2}\left(\frac{1}{m+1}\left(\frac{\partial R_m(p_1^1,p_1^2)}{\partial p_1^2}p_2^2 + \frac{\partial R_m(p_1^1,p_1^2)}{\partial p_1^1}p_2^1\right) + \right. \\ &\qquad \left. \frac{m}{m+1}\left(p_1^1p_2^2 - p_1^2p_2^1\right) S_m(p_1^1,p_1^2)\right) \\ &= \frac{1}{\Delta_p} D^i_{p_1,p_2}\left(D_{p_1,p_2} R_m(p_1^1,p_1^2) + \frac{m}{m+1}\Delta_p S_m(p_1^1,p_1^2)\right) \end{aligned} \tag{15}$$

and

$$\begin{aligned} b^2_{m-i,i} &= \frac{1}{\Delta_p} D^{m-i}_{p_2,p_1}\left(-p_1^2 P^1_m\left(p_2^1,p_2^2\right) + p_1^1 P^2_m\left(p_2^1,p_2^2\right)\right) \\ &= \frac{1}{\Delta_p} D^{m-i}_{p_2,p_1}\left(-\frac{1}{m+1}p_1^2\left(\frac{\partial R_m(p_2^1,p_2^2)}{\partial p_2^2} + m p_2^1 S_m(p_2^1,p_2^2)\right) + \right. \\ &\qquad \left. \frac{1}{m+1}p_1^1\left(-\frac{\partial R_m(p_2^1,p_2^2)}{\partial p_2^1} + m p_2^2 S_m(p_2^1,p_2^2)\right)\right) \\ &= \frac{1}{\Delta_p} D^{m-i}_{p_2,p_1}\left(\frac{1}{m+1}\left(-\frac{\partial R_m(p_2^1,p_2^2)}{\partial p_2^2}p_1^2 - \frac{\partial R_m(p_2^1,p_2^2)}{\partial p_2^1}p_1^1\right) + \right. \\ &\qquad \left. \frac{m}{m+1}\left(p_1^1p_2^2 - p_1^2p_2^1\right) S_m(p_2^1,p_2^2)\right) \\ &= \frac{1}{\Delta_p} D^{m-i}_{p_2,p_1}\left(-D_{p_2,p_1} R_m(p_2^1,p_2^2) + \frac{m}{m+1}\Delta_p S_m(p_2^1,p_2^2)\right). \end{aligned} \tag{16}$$

Note that in (15) the polynomials R_m and S_m are GL-comitants of system (1) with the characters $(m+1,\ 0;\ -1;\ 1)$ and $(m-1,\ 0;\ 0;\ 1)$, respectively, whereas in (16) the polynomials R_m and S_m are GL-comitants with the characters $(0,\ m+1;\ -1;\ 1)$ and $(0,\ m-1;\ 0;\ 1)$, respectively. Therefore, taking into consideration Property 2 and the fact that Δ_p is a $GL(2,\mathbb{R})$-comitant with the character $(1,\ 1;\ -1;\ 0)$ it follows that the numerators in the last expressions from (15) and (16) are GL-comitants with the characters $(m-i,\ i+1;\ -1;\ 1)$ and $(m-i+1,\ i;\ -1;\ 1)$, respectively. Evidently, by using index j $(j=1,2)$ these characters can be written as one formula and this leads to the character, indicated in the statement of Lemma 1.

Lemma 1 is proved.

Lemma 2. *Let $K(a,u,v)$ be a GL-comitant of the system (1) with the character $(r_1,\ r_2;\ g_K;\ d_K)$ and $U(a,w)$ and $V(a,w)$ be GL-comitants of the system (1) with the characters $(r_U;\ g_U;\ d_U)$ and $(r_V;\ g_V;\ d_V)$, respectively. If in the expression of the* *comitant $K(a,u,v)$ we replace u^1, u^2, v^1 and v^2 by the expressions $\frac{\partial U}{\partial w^2}, -\frac{\partial U}{\partial w^1}, -\frac{\partial V}{\partial w^2}$ and $\frac{\partial V}{\partial w^1}$, respectively, then we obtain the GL-comitant of system (1)*

$$\overset{*}{K}(a,w) = K\left(a, \frac{\partial U(a,w)}{\partial w^2}, -\frac{\partial U(a,w)}{\partial w^1}, -\frac{\partial V(a,w)}{\partial w^2}, \frac{\partial V(a,w)}{\partial w^1}\right)$$

with the character $(r_1(r_U-1)+r_2(r_V-1)\,;\ g_K+r_1(g_U+1)+r_2(g_V+1)\,;\ d_K+r_1d_U+r_2d_V)\,.$

Proof. Let the transformation (3) transfer the vectors u, v and w in to $\bar{u}, \bar{v}$ and $\bar{w}$, respectively. In particular, for the vector w we have

$$w^1 = p_1^1\bar{w}^1 + p_2^1\bar{w}^2, \qquad w^2 = p_1^2\bar{w}^1 + p_2^2\bar{w}^2. \tag{17}$$

So, since $K(a,u,v)$, $U(a,w)$ and $V(a,w)$ are GL-comitants of system (1), then the relations

$$\begin{gathered} K(b,\bar{u},\bar{v}) = \Delta_q^{-g_K}K(a,u,v) = \\ = \Delta_q^{-g_K}K(a,q^{-1}\cdot\bar{u},q^{-1}\cdot\bar{v}) = \Delta_q^{-g_K}K(a,p\cdot\bar{u},p\cdot\bar{v}); \\ U(b,\bar{w}) = \Delta_q^{-g_U}U(a,w); \qquad V(b,\bar{w}) = \Delta_q^{-g_V}V(a,w) \end{gathered} \tag{18}$$

hold. Taking into consideration (17) and (18) we have

$$\begin{aligned} \frac{\partial U(b,\bar{w})}{\partial \bar{w}^1} &= \Delta_q^{-g_U}\left(\frac{\partial U(a,w)}{\partial w^1}\ \frac{\partial w^1}{\partial \bar{w}^1} + \frac{\partial U(a,w)}{\partial w^2}\ \frac{\partial w^2}{\partial \bar{w}^1}\right) \\ &= \Delta_q^{-g_U}\left(\frac{\partial U(a,w)}{\partial w^1}\ p_1^1 + \frac{\partial U(a,w)}{\partial w^2}\ p_1^2\right); \\ \frac{\partial U(b,\bar{w})}{\partial \bar{w}^2} &= \Delta_q^{-g_U}\left(\frac{\partial U(a,w)}{\partial w^1}\ \frac{\partial w^1}{\partial \bar{w}^2} + \frac{\partial U(a,w)}{\partial w^2}\ \frac{\partial w^2}{\partial \bar{w}^2}\right) \\ &= \Delta_q^{-g_U}\left(\frac{\partial U(a,w)}{\partial w^1}\ p_2^1 + \frac{\partial U(a,w)}{\partial w^2}\ p_2^2\right); \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{\partial V(b,\bar{w})}{\partial \bar{w}^1} &= \Delta_q^{-g_V}\left(\frac{\partial V(a,w)}{\partial w^1}\ \frac{\partial w^1}{\partial \bar{w}^1} + \frac{\partial V(a,w)}{\partial w^2}\ \frac{\partial w^2}{\partial \bar{w}^1}\right) \\ &= \Delta_q^{-g_V}\left(\frac{\partial V(a,w)}{\partial w^1}\ p_1^1 + \frac{\partial V(a,w)}{\partial w^2}\ p_1^2\right); \\ \frac{\partial V(b,\bar{w})}{\partial \bar{w}^2} &= \Delta_q^{-g_V}\left(\frac{\partial V(a,w)}{\partial w^1}\ \frac{\partial w^1}{\partial \bar{w}^2} + \frac{\partial V(a,w)}{\partial w^2}\ \frac{\partial w^2}{\partial \bar{w}^2}\right) \\ &= \Delta_q^{-g_V}\left(\frac{\partial V(a,w)}{\partial w^1}\ p_2^1 + \frac{\partial V(a,w)}{\partial w^2}\ p_2^2\right). \end{aligned} \tag{20}$$

If we set $\bar{u}^1 = \frac{\partial U(b,\bar{w})}{\partial \bar{w}^2}$, $\bar{u}^2 = -\frac{\partial U(b,\bar{w})}{\partial \bar{w}^1}$, $\bar{v}^1 = -\frac{\partial V(b,\bar{w})}{\partial \bar{w}^2}$ and $\bar{v}^2 = \frac{\partial V(b,\bar{w})}{\partial \bar{w}^1}$, then from the first equality (18), considering (19) and (20), we obtain

$$\begin{aligned}
\overset{*}{K}(b,\bar w) &= K\left(b,\ \frac{\partial U(b,\bar w)}{\partial \bar w^2},\ -\frac{\partial U(b,\bar w)}{\partial \bar w^1},\ -\frac{\partial V(b,\bar w)}{\partial \bar w^2},\ \frac{\partial V(b,\bar w)}{\partial \bar w^1}\right)\\
&= \Delta_q^{-g_K} K\left(a,\ \frac{\partial U(b,\bar w)}{\partial \bar w^2}\ p_1^1 - \frac{\partial U(b,\bar w)}{\partial \bar w^1}\ p_2^1,\ \frac{\partial U(b,\bar w)}{\partial \bar w^2}\ p_1^2 - \frac{\partial U(b,\bar w)}{\partial \bar w^1}\ p_2^2,\right.\\
&\qquad \left.-\frac{\partial V(b,\bar w)}{\partial \bar w^2}\ p_1^1 + \frac{\partial V(b,\bar w)}{\partial \bar w^1}\ p_2^1,\ -\frac{\partial V(b,\bar w)}{\partial \bar w^2}\ p_1^2 + \frac{\partial V(b,\bar w)}{\partial \bar w^1}\ p_2^2\right)\\
&= \Delta_q^{-g_K} K\left(a,\ \Delta_q^{-g_U}\frac{\partial U(a,w)}{\partial w^2}\ \Delta_q^{-1},\ -\Delta_q^{-g_U}\frac{\partial U(a,w)}{\partial w^1}\ \Delta_q^{-1},\right.\\
&\qquad \left.-\Delta_q^{-g_V}\frac{\partial V(a,w)}{\partial w^2}\ \Delta_q^{-1},\ \Delta_q^{-g_V}\frac{\partial V(a,w)}{\partial w^1}\ \Delta_q^{-1}\right)\\
&= \Delta_q^{-g_K}\Delta_q^{-r_1(g_U+1)}\Delta_q^{-r_2(g_V+1)} K\left(a,\ \frac{\partial U(a,w)}{\partial w^2},\ -\frac{\partial U(a,w)}{\partial w^1},\right.\\
&\qquad \left.-\frac{\partial V(a,w)}{\partial w^2},\ \frac{\partial V(a,w)}{\partial w^1}\right) = \Delta_q^{-\left(g_K+r_1\left(g_U+1\right)+r_2\left(g_V+1\right)\right)}\overset{*}{K}(a,w).
\end{aligned}$$

So, the polynomial $\overset{*}{K}(a,w)$ is a $GL(2,R)$-comitant of system (1) which weight is $\left(g_K + r_1\left(g_U+1\right) + r_2\left(g_V+1\right)\right)$. Its character can easily be obtained taking into account the characters of the comitants K, U and V.

Lemma 2 is proved.

Proof of Theorem 1. The matrix of the transformation inverse to the transformation (7) has the form

$$\begin{pmatrix} p_1^1 & p_2^1\\ p_1^2 & p_2^2\end{pmatrix} = \begin{pmatrix} \frac{\partial U(a,w)}{\partial w^2} & -\frac{1}{r_V r_U (V,U)^{(1)}}\ \frac{\partial V(a,w)}{\partial w^2}\\ -\frac{\partial U(a,w)}{\partial w^1} & \frac{1}{r_V r_U (V,U)^{(1)}}\ \frac{\partial V(a,w)}{\partial w^1}\end{pmatrix}. \tag{21}$$

We observe that for the matrix (21) it occurs $\Delta_p = 1$. Then, taking into consideration the relations (10), (11) and Lemma 1, we obtain

$$\begin{aligned}
b^j_{m-i,i} &= L^j_{m-i,i}(a,p_1,p_2)\Big|_{\substack{p_1=\left(\frac{\partial U(a,w)}{\partial w^2},\ -\frac{\partial U(a,w)}{\partial w^1}\right)\\ p_2=\left(-\frac{1}{r_V r_U (V,U)^{(1)}}\ \frac{\partial V(a,w)}{\partial w^2},\ \frac{1}{r_V r_U (V,U)^{(1)}}\ \frac{\partial V(a,w)}{\partial w^1}\right)}}\\
&= \frac{1}{\left(r_V r_U (V,U)^{(1)}\right)^{i-j+2}}\ L^j_{m-i,i}(a,p_1,p_2)\Big|_{\substack{p_1=\left(\frac{\partial U(a,w)}{\partial w^2},\ -\frac{\partial U(a,w)}{\partial w^1}\right)\\ p_2=\left(-\frac{\partial V(a,w)}{\partial w^2},\ \frac{\partial V(a,w)}{\partial w^1}\right)}}\\
&= \frac{(-1)^{3-j}}{\left(r_V r_U (V,U)^{(1)}\right)^{i-j+2}}\left(p^2_{3-j} D^{m-i}_{p_2,p_1} P^1_m(p^1_2,p^2_2) - \right.\\
&\qquad \left. p^1_{3-j} D^{m-i}_{p_2,p_1} P^2_m(p^1_2,p^2_2)\right)\Big|_{\substack{p_1=\left(\frac{\partial U(a,w)}{\partial w^2},\ -\frac{\partial U(a,w)}{\partial w^1}\right)\\ p_2=\left(-\frac{\partial V(a,w)}{\partial w^2},\ \frac{\partial V(a,w)}{\partial w^1}\right)}}.
\end{aligned}$$

By Lemma 2 the numerator of the obtained expression is a GL-comitant of system (1) with the character (9) which depends on the coordinates of the vector w.

Theorem 1 is proved.

Example 1. *Assume that the GL-comitant* $K_{28}(a,w) = 3(K_4,K_5)^{(1)} \not\equiv 0$. *Applying the transformation*

$$y^1 = \frac{1}{K_{28}(a,w)}\ \frac{\partial K_4(a,w)}{\partial w^1}\ x^1 + \frac{1}{K_{28}(a,w)}\ \frac{\partial K_4(a,w)}{\partial w^2}\ x^2,$$
$$y^2 = \frac{\partial K_5(a,w)}{\partial w^1}\ x^1 + \frac{\partial K_5(a,w)}{\partial w^2}\ x^2 \tag{22}$$

the quadratic system (i.e., system (1) for $n=2$*) can be brought to the system:*

$$\frac{dy^1}{dt} = \frac{K_{27}}{K_{28}} + \frac{\frac{1}{2}K_3K_{28} - \frac{2}{3}K_5K_{18} + \frac{1}{6}K_{40}}{K_{28}}y^1 + \frac{3K_4K_{18} - K_2K_{17}}{K_{28}^2}y^2 - \frac{K_5K_{38}}{K_{28}}(y^1)^2 + \frac{3K_4K_{38} + 6K_4K_{44} - \frac{2}{3}K_5^2K_{17} + \frac{1}{3}K_5K_{36}}{K_{28}^2}y^1y^2 - \frac{K_4K_{36}}{K_{28}^3}(y^2)^2,$$
$$\frac{dy^2}{dt} = -K_{25} - K_{43}y^1 + \frac{\frac{1}{2}K_3K_{28} + \frac{2}{3}K_5K_{18} - \frac{1}{6}K_{40}}{K_{28}}y^2 - K_{52}(y^1)^2 + \frac{K_5K_{38}}{K_{28}}y^1y^2 + \frac{-\frac{3}{2}K_4K_{38} + 6K_4K_{44} - \frac{2}{3}K_5^2K_{17} - \frac{1}{6}K_5K_{36}}{K_{28}^2}(y^2)^2.$$

We note that the transformation (22) is a particular case of the transformation (7), namely, $U = K_5$ and $V = K_4$.

In the particular case when U and V are both comitants of the first degree with respect to the coordinates of the vector $w = (w^1, w^2) \in R^2$ we obtain that the GL-comitants $T^j_{m-i,i}(a,w)$ and $(V,U)^{(1)}$ are GL-invariants of system (1). Such kind of transformations are used in papers [6–8]. In paper [9] analogical transformations are applied for system (1). We shall include here two example of this type which we use below (one for quadratic and the other for cubic systems).

Example 2. *Consider the GL-invariant* $K_{52}(a) = (K_{44},K_5)^{(1)} \neq 0$. *By using the transformation*

$$y^1 = \frac{1}{K_{52}(a)}\ \frac{\partial K_{44}(a,w)}{\partial w^1}\ x^1 + \frac{1}{K_{52}(a)}\ \frac{\partial K_{44}(a,w)}{\partial w^2}\ x^2,$$
$$y^2 = \frac{\partial K_5(a,w)}{\partial w^1}\ x^1 + \frac{\partial K_5(a,w)}{\partial w^2}\ x^2$$

quadratic system can be brought to the system:

$$\frac{dy^1}{dt} = \frac{K_{51}}{K_{52}} + \frac{\frac{1}{2}K_3K_{52} + \frac{1}{2}K_{53}}{K_{52}}y^1 + \frac{\frac{1}{3}K_{52}K_{39} - \frac{1}{3}K_{43}K_{48}}{K_{52}^2}y^2 + \frac{\frac{2}{3}K_{52} + \frac{2}{3}K_{48}}{K_{52}}y^1y^2 - \frac{K_{59}}{K_{52}^2}(y^2)^2, \tag{23}$$

$$\frac{dy^2}{dt} = -K_{25} - K_{43}y^1 + \frac{\frac{1}{2}K_3K_{52} - \frac{1}{2}K_{53}}{K_{52}}y^2 - K_{52}(y^1)^2 + \frac{\frac{2}{3}K_{52} - \frac{1}{3}K_{48}}{K_{52}}(y^2)^2.$$

Example 3. *We consider the GL-invariant $K_{25} = -(K_5, K_1)^{(1)} \neq 0$. Applying the transformation*

$$y^1 = -\frac{1}{K_{25}(a)}\ \frac{\partial K_5(a,w)}{\partial w^1}\ x^1 - \frac{1}{K_{25}(a)}\ \frac{\partial K_5(a,w)}{\partial w^2}\ x^2,$$
$$y^2 = \frac{\partial K_1(a,w)}{\partial w^1}\ x^1 + \frac{\partial K_1(a,w)}{\partial w^2}\ x^2$$

cubic system (i.e., system (1) for $n = 3$) can be brought to the system:

$$\frac{dy^1}{dt} = 1 + \frac{\frac{1}{2}K_3K_{25} + \frac{1}{2}K_{42}}{K_{25}}y^1 + \frac{K_{43}}{K_{25}^2}y^2 - \frac{\frac{2}{3}K_{25}^2 + \frac{1}{3}K_{50}}{K_{25}}(y^1)^2 + \frac{2K_{51}}{K_{25}^2}y^1y^2 +$$
$$\frac{K_{52}}{K_{25}^3}(y^2)^2 + \frac{\frac{3}{4}K_{25}K_{45} + \frac{1}{4}K_{55}}{K_{25}}(y^1)^3 + \frac{\frac{3}{2}K_{25}K_{46} + \frac{3}{2}K_{56}}{K_{25}^2}(y^1)^2y^2 +$$
$$\frac{\frac{3}{4}K_{25}K_{47} + \frac{3}{4}K_{57}}{K_{25}^3}y^1(y^2)^2 + \frac{K_{58}}{K_{25}^4}(y^2)^3,$$
$$\frac{dy^2}{dt} = -K_{41}y^1 + \frac{\frac{1}{2}K_3K_{25} - \frac{1}{2}K_{42}}{K_{25}}y^2 - K_{49}(y^1)^2 - \frac{\frac{2}{3}K_{25}^2 + \frac{2}{3}K_{50}}{K_{25}}y^1y^2 -$$
$$\frac{K_{51}}{K_{25}^2}(y^2)^2 - K_{54}(y^1)^3 + \frac{\frac{3}{4}K_{25}K_{45} - \frac{3}{4}K_{55}}{K_{25}}(y^1)^2y^2 +$$
$$\frac{\frac{3}{2}K_{25}K_{46} - \frac{3}{2}K_{56}}{K_{25}^2}y^1(y^2)^2 + \frac{\frac{3}{4}K_{25}K_{47} - \frac{1}{4}K_{57}}{K_{25}^3}(y^2)^3. \tag{24}$$

3 Transformation by using only one $GL(2,\mathbb{R})$-comitant

Theorem 2. *Let $V(a,w) \not\equiv 0$ be a GL-comitant of system (1) with the character $(r_V; g_V; d_V)$. By using the transformation*

$$y^1 = \frac{1}{r_V V(a,w)}\ \frac{\partial V(a,w)}{\partial w^1}\ x^1 + \frac{1}{r_V V(a,w)}\ \frac{\partial V(a,w)}{\partial w^2}\ x^2,$$
$$y^2 = -w^2x^1 + w^1x^2 \tag{25}$$

system (1) can be brought to the system (4) with the coefficients

$$b^j_{m-i,i} = \frac{T^j_{m-i,i}(a,w)}{(r_V V(a,w))^{i-j+2}}, \quad (j = 1,2;\ m = 0,\cdots,n;\ i = 0,\cdots,m), \tag{26}$$

where

$$T^j_{m-i,i}(a,w) = (-1)^{3-j}\left(p^2_{3-j}D^{m-i}_{p_2,p_1}P^1_m(p^1_2,p^2_2) - p^1_{3-j}D^{m-i}_{p_2,p_1}P^2_m(p^1_2,p^2_2)\right)\Bigg|\begin{array}{l} p_1 = (w^1,\ w^2) \\ p_2 = \left(-\frac{\partial V(a,w)}{\partial w^2},\ \frac{\partial V(a,w)}{\partial w^1}\right)\end{array}$$

is a GL-comitant of system (1) with the character

$$(m-i+j-1+(i-j+2)(r_V-1);\ -1+(i-j+2)(g_V+1);\ 1+(i-j+2)d_V\,). \qquad (27)$$

Lemma 3. *Let $K(a,u,v)$ be a GL-comitant of the system (1) with the character $(r_1,\ r_2;\ g_K;\ d_K)$ and let $V(a,w)$ be a GL-comitant of the system (1) with the character $(r_V;\ g_V;\ d_V)$. If in the expression of the comitant $K(a,u,v)$ we replace u^1,u^2,v^1 and v^2 by the expressions $w^1,w^2,-\frac{\partial V}{\partial w^2}$ and $\frac{\partial V}{\partial w^1}$, respectively, then we obtain the GL-comitant*

$$\overset{*}{K}(a,w)=K\left(a,\ w^1,\ w^2,\ -\frac{\partial V(a,w)}{\partial w^2},\ \frac{\partial V(a,w)}{\partial w^1}\right)$$

of system (1) with the character

$$(r_1+r_2\,(r_V-1)\,;\ g_K+r_2\,(g_V+1)\,;\ d_K+r_2 d_V\,)\,.$$

Proof. Let the transformation (3) transfer the vectors u,v and w in to $\bar u,\bar v$ and $\bar w$, respectively. In particular, for the vector w the relations (17) hold. Since the polynomials $K(a,u,v)$ and $V(a,w)$ are GL-comitants of system (1) the relations (18) are fulfilled. Then the relations (17) and (18) yield (20).

By setting $\bar u^1=\bar w^1,\ \bar u^2=\bar w^2,\ \bar v^1=-\frac{\partial V(b,\bar w)}{\partial \bar w^2}$ and $\bar v^2=\frac{\partial V(b,\bar w)}{\partial \bar w^1}$, taking into account (20) from the first equality (18) we obtain

$$\begin{aligned}\overset{*}{K}(b,\bar w)&=K\left(b,\ \bar w^1,\ \bar w^2,\ -\frac{\partial V(b,\bar w)}{\partial \bar w^2},\ \frac{\partial V(b,\bar w)}{\partial \bar w^1}\right)\\ &=\Delta_q^{-g_K}K\Big(a,\ \bar w^1\ p_1^1+\bar w^2\ p_2^1,\ \bar w^1\ p_1^2+\bar w^2\ p_2^2,\\ &\qquad -\frac{\partial V(b,\bar w)}{\partial \bar w^2}\ p_1^1+\frac{\partial V(b,\bar w)}{\partial \bar w^1}\ p_2^1,\ -\frac{\partial V(b,\bar w)}{\partial \bar w^2}\ p_1^2+\frac{\partial V(b,\bar w)}{\partial \bar w^1}\ p_2^2\Big)\\ &=\Delta_q^{-g_K}K\left(a,\ w^1,\ w^2,\ -\Delta_q^{-g_V}\ \frac{\partial V(a,w)}{\partial w^2}\ \Delta_q^{-1},\ \Delta_q^{-g_V}\ \frac{\partial V(a,w)}{\partial w^1}\ \Delta_q^{-1}\right)\\ &=\Delta_q^{-g_K}\Delta_q^{-r_2(g_V+1)}K\left(a,\ w^1,\ w^2,\ -\frac{\partial V(a,w)}{\partial w^2},\ \frac{\partial V(a,w)}{\partial w^1}\right)\\ &=\Delta_q^{-\left(g_K+r_2\left(g_V+1\right)\right)}\overset{*}{K}(a,w).\end{aligned}$$

Thus the polynomial $\overset{*}{K}(a,w)$ is a GL-comitant of system (1) with the weight $g_K+r_2\,(g_V+1)$. Its character can be easily determined by the characters of the comitants K and V. Lemma 3 is proved.

Taking into consideration Lemma 3 the proof of Theorem 2 is analogous to the proof of Theorem 1.

Example 4. *Assume that the GL-comitant $K_4(a,w)\not\equiv 0$. Then the transformation*

$$y^1=\frac{1}{3K_4(a,w)}\ \frac{\partial K_4(a,w)}{\partial w^1}\ x^1+\frac{1}{3K_4(a,w)}\ \frac{\partial K_4(a,w)}{\partial w^2}\ x^2,$$

$$y^2 = -w^2x^1 + w^1x^2$$

will bring generic quadratic system to the system:

$$\begin{aligned}\frac{dy^1}{dt} &= \frac{K_{27}}{3K_4} + \frac{\frac{3}{2}K_3K_4 + \frac{1}{2}K_{19}}{3K_4}y^1 + \frac{3K_4K_{18} - K_2K_{17}}{9K_4^2}y^2 + \\ &\quad \frac{1}{3}K_5(y^1)^2 + \frac{2K_{28} + 2K_{17}}{9K_4}y^1y^2 - \frac{K_{36}}{27K_4^2}(y^2)^2, \\ \frac{dy^2}{dt} &= -K_1 - K_2y^1 + \frac{\frac{3}{2}K_3K_4 - \frac{1}{2}K_{19}}{3K_4}y^2 - K_4(y^1)^2 + \\ &\quad \frac{1}{3}K_5y^1y^2 + \frac{2K_{28} - K_{17}}{9K_4}(y^2)^2. \end{aligned} \tag{28}$$

Example 5. *Suppose that the GL-comitant $K_5(a, w) \not\equiv 0$. Applying the transformation*

$$\begin{aligned} y^1 &= \frac{1}{K_5(a,w)}\ \frac{\partial K_5(a,w)}{\partial w^1}\ x^1 + \frac{1}{K_5(a,w)}\ \frac{\partial K_5(a,w)}{\partial w^2}\ x^2, \\ y^2 &= -w^2x^1 + w^1x^2 \end{aligned}$$

quadratic system of differential equations will be transformed in to the system:

$$\begin{aligned}\frac{dy^1}{dt} &= -\frac{K_{25}}{K_5} + \frac{\frac{1}{2}K_3K_5 - \frac{1}{2}K_{26}}{K_5}y^1 + \frac{K_{43}}{K_5^2}y^2 + \\ &\quad \frac{\frac{2}{3}K_5^2 - \frac{1}{3}K_{28}}{K_5}(y^1)^2 + \frac{2K_{44}}{K_5^2}y^1y^2 - \frac{K_{52}}{K_5^3}(y^2)^2, \\ \frac{dy^2}{dt} &= -K_1 - K_2y^1 + \frac{\frac{1}{2}K_3K_5 + \frac{1}{2}K_{26}}{K_5}y^2 - K_4(y^1)^2 + \\ &\quad \frac{\frac{2}{3}K_5^2 + \frac{2}{3}K_{28}}{K_5}y^1y^2 - \frac{K_{44}}{K_5^2}(y^2)^2. \end{aligned} \tag{29}$$

Since for $K_4(a, w) \equiv 0 \equiv K_5(a, w)$ the quadratic homogeneous parts of quadratic system vanish, we conclude that any quadratic system can be transformed in to the canonical system either (28) or (29).

Example 6. *Let the condition $K_6(a, w) \not\equiv 0$ hold. By means of the transformation*

$$\begin{aligned} y^1 &= \frac{1}{4K_6(a,w)}\ \frac{\partial K_6(a,w)}{\partial w^1}\ x^1 + \frac{1}{4K_6(a,w)}\ \frac{\partial K_6(a,w)}{\partial w^2}\ x^2, \\ y^2 &= -w^2x^1 + w^1x^2 \end{aligned}$$

the cubic system can be transformed in to the system:

$$\frac{dy^1}{dt} = \frac{K_{29}}{4K_6} + \frac{2K_3K_6 + K_{14}}{4K_6}y^1 + \frac{4K_6K_{13} - K_2K_9}{16K_6^2}y^2 +$$

$$\frac{\frac{8}{3}K_5K_6+\frac{1}{3}K_{12}}{4K_6}(y^1)^2+\frac{\frac{8}{3}K_6K_{30}-2K_4K_{14}+\frac{16}{3}K_6K_{11}}{16K_6^2}y^1y^2+$$
$$\frac{16K_6^2K_{10}-K_9K_{12}-2K_4K_{33}}{64K_6^3}(y^2)^2+\frac{3}{4}K_7(y^1)^3+$$
$$\frac{6K_{16}+3K_9}{16K_6}(y^1)^2y^2+\frac{48K_6K_{15}-3K_7K_9-6K_{33}}{64K_6^2}y^1(y^2)^2+$$
$$\frac{64K_6^2K_8-3K_9^2}{256K_6^3}(y^2)^3,$$
$$\frac{dy^2}{dt}=-K_1-K_2y^1+\frac{2K_3K_6-K_{14}}{4K_6}y^2-K_4(y^1)^2+\frac{\frac{8}{3}K_5K_6-\frac{2}{3}K_{12}}{4K_6}y^1y^2+$$
$$\frac{\frac{8}{3}K_6K_{30}+K_4K_{14}-\frac{8}{3}K_6K_{11}}{16K_6^2}(y^2)^2-K_6(y^1)^3+\frac{3}{4}K_7(y^1)^2y^2+$$
$$\frac{6K_{16}-3K_9}{16K_6}y^1(y^2)^2+\frac{48K_6K_{15}-3K_7K_9+2K_{33}}{64K_6^3}(y^2)^3. \qquad (30)$$

Example 7. *Assume that the GL-comitant $K_7(a,w))\not\equiv 0$. Applying the transformation*

$$y^1=\frac{1}{2K_7(a,w)}\ \frac{\partial K_7(a,w)}{\partial w^1}\ x^1+\frac{1}{2K_7(a,w)}\ \frac{\partial K_7(a,w)}{\partial w^2}\ x^2,$$
$$y^2=-w^2x^1+w^1x^2$$

the generic cubic system can be brought to the following system:

$$\frac{dy^1}{dt}=\frac{K_{31}}{K_7}+\frac{K_3K_7-K_{23}}{2K_7}y^1+\frac{4K_7K_{22}-4K_2K_{24}}{4K_7^2}y^2+$$
$$\frac{\frac{4}{3}K_5K_7-\frac{2}{3}K_{21}}{2K_7}(y^1)^2+\frac{\frac{8}{3}K_7K_{32}-8K_4K_{24}+\frac{8}{3}K_7K_{20}}{4K_7^2}y^1y^2+$$
$$\frac{\frac{8}{3}K_{21}K_{24}-\frac{8}{3}K_7K_{37}}{8K_7^3}(y^2)^2+\frac{\frac{3}{2}K_7^2-\frac{1}{2}K_{16}}{2K_7}(y^1)^3+$$
$$\frac{12K_7K_{15}-12K_6K_{24}}{4K_7^2}(y^1)^2y^2+$$
$$\frac{6K_7^2K_{24}-24K_7K_{35}+6K_{16}K_{24}}{8K_7^3}y^1(y^2)^2+$$
$$\frac{K_6K_{24}^2+16K_7^2K_{34}-32K_7K_{15}K_{24}}{16K_7^4}(y^2)^3,$$
$$\frac{dy^2}{dt}=-K_1-K_2y^1+\frac{K_3K_7+K_{23}}{2K_7}y^2-K_4(y^1)^2+\frac{\frac{4}{3}K_5K_7+\frac{4}{3}K_{21}}{2K_7}y^1y^2+$$
$$\frac{\frac{8}{3}K_7K_{32}+4K_4K_{24}-\frac{4}{3}K_7K_{20}}{4K_7^2}(y^2)^2-K_6(y^1)^3+\frac{\frac{3}{2}K_7^2+\frac{3}{2}K_{16}}{2K_7}(y^1)^2y^2+$$
$$\frac{12K_6K_{24}-12K_7K_{15}}{4K_7^2}y^1(y^2)^2+\frac{6K_7^2K_{24}+8K_7K_{35}-2K_{16}K_{24}}{8K_7^3}(y^2)^3. \qquad (31)$$

Since for $K_6(a,w) \equiv 0 \equiv K_7(a,w)$ the cubic homogeneous parts of generic cubic system vanish, we conclude that any cubic system can be transformed in to the canonical system either (30) or (31).

4 Rational bases of $GL(2,\mathbb{R})$-comitants

Let $K(a)$ be a GL-invariant of system (1) with the character $(0;\ g;\ d)$ and let $K(b)$ be the same invariant calculated for transformed system (1) via the transformation (7) with two GL-comitants U and V of the first degree with respect to the coordinates of the vector w. Observe that for the matrix q of the transformation (7) we have $\Delta_q = 1$. Since $K(a)$ is a GL-invariant of system (1) with the character $(0;\ g;\ d)$, by Definition 1 the following relation holds: $K(b) = \Delta_q^{-g} K(a) = K(a)$. Hence, by Theorem 1 it follows that any GL-invariant of system (1) will be a polynomial of the expressions

$$b^j_{m-i,i} = \frac{T^j_{m-i,i}(a)}{\left((V,U)^{(1)}\right)^{i-j+2}} \quad (j = 1,2;\ m = 0,\cdots,n;\ i = 0,\cdots,m).$$

As result we obtain a rational function nominator of which is a polynomial of the invariants

$$\left\{ (V,U)^{(1)}(a);\ T^j_{m-i,i}(a)\ \middle|\ j = 1,2;\ m = 0,\cdots,n;\ i = 0,\cdots,m \right\}, \qquad (32)$$

whereas its denominator is a nonnegative integer power of the GL-invariant $(V,U)^{(1)}(a)$.

Thus, we obtain the next result.

Theorem 3. *If the GL-comitants $U(a,w)$ and $V(a,w)$ from transformation (7) have the first degree with respect to the coordinates of the vector $w \in R^2$ and $(V,U)^{(1)} \neq 0$, then the set of invariants (32) is a rational basis of the GL-invariants of system (1) on $M = \left\{ a \in A \mid (V,U)^{(1)} \neq 0 \right\}$.*

By Example 2 and Theorem 3 we obtain the following theorem.

Theorem 4. *The set of GL-invariants*

$$\{K_{52},\ K_3,\ K_{25},\ K_{39},\ K_{43},\ K_{51},\ K_{48},\ K_{53},\ K_{59}\} \qquad (33)$$

is a rational basis of GL-invariants of quadratic system on $M = \{ a \in A \mid K_{52} \neq 0 \}$.

Analogously, from Example 3 and Theorem 3 we obtain for cubic system the next result.

Theorem 5. *The set of GL-invariants*

$$\begin{gathered}\{K_{25},\ K_3,\ K_{41},\ K_{42},\ K_{43},\ K_{45},\ K_{46},\ K_{47},\ K_{49},\\ K_{50},\ K_{51},\ K_{52},\ K_{54},\ K_{55},\ K_{56},\ K_{57},\ K_{58}\}\end{gathered} \qquad (34)$$

is a rational basis of GL-invariants of cubic system on $M = \{ a \in A \mid K_{25} \neq 0 \}$.

In what follows we shall use the following lemma.

Lemma 4. *Let $K(a,u)$ be a GL–comitant of system (1) with the character $(r,\ g,\ d)$, q be the matrix of transformation (25) and $v = q \cdot u$. Assume that $K\left(b\left(w^1, w^2\right), v\right)$ is the comitant K calculated for system (4) which is obtained from system (1) via transformation (25). Then the following identity holds:*

$$K(a,u) = \frac{1}{r!}\left.\frac{\partial^r K\left(b\left(w^1,w^2\right),v\right)}{\partial\left(v^1\right)^r}\right|_{\substack{w^1=u^1\\ w^2=u^2}}. \tag{35}$$

Proof. Observe that for the matrix q of the transformation (25) we have $\Delta_q = 1$ and

$$\begin{aligned} u^1 &= w^1 v^1 - \frac{1}{r_V r_U (V,U)^{(1)}}\,\frac{\partial V(a,w)}{\partial w^2} v^2,\\ u^2 &= w^2 v^1 + \frac{1}{r_V r_U (V,U)^{(1)}}\,\frac{\partial V(a,w)}{\partial w^1} v^2. \end{aligned}$$

Since $K(a,u)$ is a GL-comitant of system (1) with the character $(r,\ g,\ d)$ by Definition 1 the following equalities are valid:

$$\begin{aligned} \frac{1}{r!}\frac{\partial^r K\left(b\left(w^1,w^2\right),v\right)}{\partial\left(v^1\right)^r} &= \frac{1}{r!}\Delta_q^{-g}\frac{\partial^r K(a,u)}{\partial\left(v^1\right)^r}\\ = \frac{1}{r!}\Delta_q^{-g}\sum_{i=0}^{r}\binom{r}{i}&\frac{\partial^r K(a,u)}{\partial\left(u^1\right)^{r-i}\partial\left(u^2\right)^i}\left(\frac{\partial u^1}{\partial v^1}\right)^{r-i}\left(\frac{\partial u^2}{\partial v^1}\right)^i\\ = \frac{1}{r!}\Delta_q^{-g}\sum_{i=0}^{r}\binom{r}{i}&\frac{\partial^r K(a,u)}{\partial\left(u^1\right)^{r-i}\partial\left(u^2\right)^i}\left(w^1\right)^{r-i}\left(w^2\right)^i\\ = \Delta_q^{-g} D_{uw}^r K(a,u) &= \Delta_q^{-g} K(a,w) = K(a,w). \end{aligned}$$

Consequently, the identity (35) holds.

Lemma 4 is proved.

According to Lemma 4 and Theorem 2 it follows that any GL-comitant $K(a,u)$ of system (1) is a polynomial of the expressions:

$$b_{m-i,i}^j = \frac{T_{m-i,i}^j(a,w)}{\left(V(a,w)\right)^{i-j+2}} \quad (j=1,2;\ m=0,\cdots,n;\ i=0,\cdots,m).$$

Consequently, $K(a,u)$ is a rational function nominator of which is a polynomial of the GL-comiatants

$$\left\{V(a,w);\ T_{m-i,i}^j(a,w)\ \middle|\ j=1,2;\ m=0,\cdots,n;\ i=0,\cdots,m\right\}, \tag{36}$$

whereas its denominator is a nonnegative integer power of the GL-comitant $V(a,w)$.

Thus, the following theorem is proved.

Theorem 6. *Consider a GL-comitant $V(a,w) \not\equiv 0$ of system (1). Then the set (36) of the comitants obtained via the transformation (25) of system (1) is a rational basis of the GL-comitants of system (1) on $M = \{a \in A \mid V(a,w) \not\equiv 0\}$.*

By Examples 4 and 5 and Theorem 6 for quadratic system the following result is valid.

Theorem 7. *1) The set of GL-comitants*

$$\{K_4,\ K_1,\ K_2,\ K_3,\ K_5,\ K_{17},\ K_{18},\ K_{19},\ K_{27},\ K_{28},\ K_{36}\} \tag{37}$$

is a rational basis of the GL-comitants of quadratic system on

$$M = \{a \in A \mid K_4(a,w) \not\equiv 0\}.$$

2) The set of GL-comitants

$$\{K_5,\ K_1,\ K_2,\ K_3,\ K_4,\ K_{25},\ K_{26},\ K_{28},\ K_{43},\ K_{44},\ K_{52}\} \tag{38}$$

is a rational basis of the GL-comitants of quadratic system on

$$M = \{a \in A \mid K_5(a,w) \not\equiv 0\}.$$

From Examples 6 and 7 by Theorem 6 for cubic system we obtain the next result.

Theorem 8. *1) The set of GL-comitants*

$$\begin{gathered}\{K_6,\ K_1,\ K_2,\ K_3,\ K_4,\ K_5,\ K_7,\ K_8,\ K_9,\ K_{10},\\ K_{11},\ K_{12},\ K_{13},\ K_{14},\ K_{15},\ K_{16},\ K_{29},\ K_{30},\ K_{33}\}\end{gathered} \tag{39}$$

is a rational basis of the GL-comitants of cubic system on $M = \{a \in A \mid K_6 \not\equiv 0\}$.

2) The set of GL-comitants

$$\begin{gathered}\{K_7,\ K_1,\ K_2,\ K_3,\ K_4,\ K_5,\ K_6,\ K_{15},\ K_{16},\ K_{20},\\ K_{21},\ K_{22},\ K_{23},\ K_{24},\ K_{31},\ K_{32},\ K_{34},\ K_{35},\ K_{37}\}\end{gathered} \tag{40}$$

is a rational basis of the GL-comitants of cubic system on $M = \{a \in A \mid K_7 \not\equiv 0\}$.

Remark 1. *1) The rational bases (33) and (34) of GL-invariants of quadratic and cubic systems, respectively, are minimal, i.e. their elements are polynomially independent.*

2) The rational bases (37) and (38) of GL-comitants of the quadratic system are minimal.

3) The rational bases (39) and (40) of GL-comitants of the cubic system are minimal.

The statement of Theorem 8 was published in [10]. Using Theorem 8 the independent syzygies of GL-comitants for homogeneous cubic system of differential equations are established in paper [11].

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