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About some equations of the third order with six poles

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Abstract. Investigating ordinary differential equations of the third order on the subject of belonging to P-type (solutions of such equations have no movable critical singular points), Chazy has built an equation (Chazy equation) with 32 coefficients. If these coefficients satisfy the special (S)-system, then Chazy equation belongs to P-type. In this paper we find three solution of the (S)-system and build three classes of Chazy equation of the P-type.

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Having researched nonlinear differential equations of the third order on the subject of belonging to P-type (solutions of such equations have no movable critical singular points), Chazy have obtained the equation [1]

$$w''' = \sum_{k=1}^{6} \frac{(w'-a_k')(w''-a_k'') + A_k(w'-a_k')^3 + B_k(w'-a_k')^2 + C_k(w'-a_k')}{w-a_k} +$$

$$+Dw'' + Ew' + \prod_{i=1}^{6}(w-a_i)\sum_{k=1}^{6}\frac{F_k}{w-a_k}, \tag{1}$$

32 coefficients of equation (1) $A_k, B_k, C_k, F_k, D, E, a_k$ $(k=\overline{1,6})$ are functions of z.

The aim of this paper is building of three classes of equations (1) of P-type.

Equation (1) is connected quite closely with Painleve equations [2]. Investigation of equation (1) is also connected with the theory of isomonodromy deformation of linear systems, the theory of golonomic quantum fields and nonlinear evolution equations. The necessary and sufficient conditions of belonging of equation (1) to P-type are a system (S) [1], which consists of 31 algebraic and differential equations

$$\sum_{k=1}^{6} A_k = 0, \quad \sum_{k=1}^{6} a_k A_k = 0, \quad \sum_{k=1}^{6} a_k^2 A_k = 0, \tag{2}$$

$$2A_k^2 + \sum_j \frac{A_k - A_j}{a_k - a_j} = 0 \quad (k, j = \overline{1,6};\ j \neq k), \tag{3}$$

$$2D+\sum_{k=1}^{6}(B_k-3a_k'A_k)=0,\quad \sum_{k=1}^{6}F_k=\sum_{k=1}^{6}a_kF_k=\sum_{k=1}^{6}a_k^2F_k=0,\tag{4}$$

$$-(\frac{5}{2}A_k+\sum_j\frac{1}{a_k-a_j})B_k+\sum_j(\frac{1}{2}A_k+\frac{1}{a_k-a_j})B_j=-A_k'+$$

$$+A_k\sum_j\frac{a_k'-a_j'}{a_k-a_j}-3\sum_j A_j\frac{a_k'-a_j'}{a_k-a_j}+\frac{3}{2}A_k\sum_{i=1}^{6}a_i'A_i,\tag{5}$$

$$-(2A_k+\sum_j\frac{1}{a_k-a_j})C_k+\sum_j C_j\frac{1}{a_k-a_j}=B_k^2-B_k'-B_k\sum_j\frac{a_k'-a_j'}{a_k-a_j}-$$

$$-\sum_j\frac{3A_j(a_k'-a_j')^2+2B_j(a_k'-a_j')}{a_k-a_j}+B_kD-E-\sum_j\frac{a_k''-a_j''}{a_k-a_j},\tag{6}$$

$$-a_k'''-B_kC_k+C_k'+\sum_j\frac{(a_k'-a_j')(a_k''-a_j''-C_k)+A_j(a_k'-a_j')^3}{a_k-a_j}+$$

$$+\sum_j\frac{B_j(a_k'-a_j')^2+C_j(a_k'-a_j')}{a_k-a_j}+E\ a_k'+D(a_k''-C_k)+F_k\ \prod_j(a_k-a_j)=0,\tag{7}$$

where $k,j=\overline{1,6};\ j\neq k$.

Chasy did not investigate the (S) system and therefore did not single out explicitly the classes of equations like (1), which are P-type equations. Prof. N.A. Lukashevich continued the investigation of system (S). In [3] he proved that solution of systems (2), (3) is

$$A_k=-1/a_k\quad (k=\overline{1,6}).\tag{8}$$

The search of solutions of systems (4)-(7) is contained in the papers [4, 5]. Here to simplify calculations we consider the case when a_k $(k=\overline{1,6})$ are constants.

Let us consider system (5). From the relation (4_1) (the first relation of the system (4)) we find

$$D=\sum_{i=1}^{6}(-\frac{1}{2}\ B_i+\frac{3}{2}a_i'A_i).\tag{9}$$

Using relation (9) we rewrite system (5) as

$$-3A_kB_k+\sum_j\frac{B_j-B_k}{a_k-a_j}=-A_k'+\sum_j(A_k-3A_j)\frac{a_k'-a_j'}{a_k-a_j}+A_kD\quad (k,j=\overline{1,6};\ j\neq k).\tag{10}$$

We simplify the sum in the right-hand side of (10) (here we use relations (8))

$$\sum_j \left(-\frac{1}{a_k}+\frac{3}{a_j}\right)\ \frac{a_k'-a_j'}{a_k-a_j}=-\frac{1}{a_k}\ \sum_j \frac{a_k'-a_j'}{a_k-a_j}+3\sum_j \frac{1}{a_j}\frac{a_k'-a_j'}{a_k-a_j}=$$

$$=\frac{2}{a_k}\ \sum_j \frac{a_k'-a_j'}{a_k-a_j}+\frac{3}{a_k}\left(a_k'\sum_j \frac{1}{a_j}-\sum_j \frac{a_j'}{a_j}\right). \tag{11}$$

From equalities (8) it follows that

$$\sigma_1=\sigma_5=0, \tag{12}$$

where $\sigma_1,\ \sigma_5$ are the first and the fifth elementary symmetric polynomials composed of the elements a_k $(k=\overline{1,6})$. Using (12) we get $\sum_j \frac{1}{a_j}=\sum_{i=1}^{6}\frac{1}{a_i}-\frac{1}{a_k}=-\frac{1}{a_k}$. Then expression (11) is

$$\frac{2}{a_k}\sum_j \frac{a_k'-a_j'}{a_k-a_j}-3\frac{a_k'}{a_k^2}-\frac{3}{a_k}\sum_j \frac{a_j'}{a_j}. \tag{13}$$

Using (13) we can write system (10) in the form

$$\frac{3}{a_k}B_k+\frac{B_j-B_k}{a_k-a_j}=-\frac{a_k'}{a_k^2}+\frac{2}{a_k}\sum_j \frac{a_k'-a_j'}{a_k-a_j}-3\frac{a_k'}{a_k^2}-\frac{3}{a_k}\frac{\sigma_6'}{\sigma_6}-\frac{D}{a_k}\quad (k,j=\overline{1,6};\ j\neq k), \tag{14}$$

where $\sigma_6=\prod_{i=1}^{6}a_i$. Let us set

$$B_i=\psi_i-\frac{1}{3}D-2\frac{a_i'}{a_i}-\frac{1}{3}\ \frac{\sigma_6'}{\sigma_6}\quad (i=\overline{1,6}). \tag{15}$$

Then system (15) is

$$\left(\frac{3}{a_k}-\sum_j \frac{1}{a_k-a_j}\right)\ \psi_k+\sum_j \frac{\psi_j}{a_k-a_j}=-3\ (a_k^{-1})'\quad (k,j=\overline{1,6};\ j\neq k), \tag{16}$$

where ψ_k $(k=\overline{1,6})$ are unknown values. A simple calculation shows us that the determinant of the system (16) is equal to zero. Hence according to Kroneker-Capelli criterion for the compatibility of system (16) it is necessary and sufficient that the rank of extended matrix be equal to the rank of matrix of system (16). This condition is true if a_k $(k=\overline{1,6})$ are constants. Then the system (16) has the form

$$\left(\frac{3}{a_k}-\sum_j \frac{1}{a_k-a_j}\right)\ \psi_k+\sum_j \frac{\psi_j}{a_k-a_j}=0\quad (k,j=\overline{1,6};\ j\neq k).$$

Solving this system we obtain

$$\psi_1=\frac{a_1}{a_6}\ \frac{a_2a_3+a_2a_4+a_2a_5+a_2a_6+a_3a_4+a_3a_5+a_3a_6+a_4a_5+a_4a_6+a_5a_6}{a_1a_2+a_1a_3+a_1a_4+a_1a_5+a_2a_3+a_2a_4+a_2a_5+a_3a_4+a_3a_5+a_4a_5}\ \psi_6,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{17}$$

$$\psi_5 = \frac{a_5}{a_6}\,\frac{a_1a_2+a_1a_3+a_1a_4+a_1a_6+a_2a_3+a_2a_4+a_2a_6+a_3a_4+a_3a_6+a_4a_6}{a_1a_2+a_1a_3+a_1a_4+a_1a_5+a_2a_3+a_2a_4+a_2a_5+a_3a_4+a_3a_5+a_4a_5}\,\psi_6.$$

Using elementary symmetric polynomials we rewrite the relations (17) as

$$\psi_i = \frac{\delta_i}{\delta_6}\,\psi_6 \quad (i=\overline{1,5}),$$

where

$$\delta_k = a_k\,(\sigma_2 + a_k^2) \quad (k=\overline{1,6}), \tag{18}$$

ψ_6 is an arbitrary analytical function of z.

Let us set

$$\psi \equiv \psi_k/\delta_k \quad (k=\overline{1,6}). \tag{19}$$

Using the substitution (18), (19), we obtain a solution of the system (5) in the form

$$B_k = a_k(\sigma_2 + a_k^2)\psi - \frac{1}{3}\,D \quad (k=\overline{1,6}), \tag{20}$$

where D is a known function, ψ is an arbitrary analytical function of z. Further we shall use the relations [5]

$$\sum_{k=1}^{6}\frac{a_k^n}{\phi(a_k)} \equiv 0 \quad (n=\overline{0,4}),\quad \sum_{k=1}^{6}\frac{a_k^5}{\phi(a_k)} \equiv 1,\quad \sum_{k=1}^{6}\frac{a_k^6}{\phi(a_k)} \equiv \sigma_1,$$

$$\sum_{k=1}^{6}\frac{a_k^7}{\phi(a_k)} \equiv \sigma_1^2 - \sigma_2,\quad \sum_{k=1}^{6}\frac{a_k^8}{\phi(a_k)} \equiv \sigma_1^3 - 2\sigma_1\sigma_2 + \sigma_3, \tag{21}$$

where $\phi(a_k) \equiv \prod_j\,(a_k - a_j)$ $(j \neq k; j,k=\overline{1,6})$, σ_k $(k=\overline{1,6})$ is an elementary symmetric polynomial composed of the elements a_k $(k=\overline{1,6})$. For $\sigma_1=\sigma_5=0$ from the identities (21) we have

$$\sum_{k=1}^{6}\frac{a_k^n}{\phi(a_k)} \equiv 0 \quad (n=\overline{0,4},6),\quad \sum_{k=1}^{6}\frac{a_k^5}{\phi(a_k)} \equiv 1,\quad \sum_{k=1}^{6}\frac{a_k^7}{\phi(a_k)} \equiv -\sigma_2,$$

$$\sum_{k=1}^{6}\frac{a_k^8}{\phi(a_k)} \equiv \sigma_3,\quad \sum_{k=1}^{6}\frac{a_k^9}{\phi(a_k)} \equiv \sigma_2^2 - \sigma_4,\quad \sum_{k=1}^{6}\frac{a_k^{10}}{\phi(a_k)} \equiv (-2)\sigma_2\sigma_3, \tag{22}$$

$$\sum_{k=1}^{6}\frac{a_k^{11}}{\phi(a_k)} \equiv \sigma_3^2 - \sigma_2^3 + 2\sigma_2\sigma_4 - \sigma_6,\quad \sum_{k=1}^{6}\frac{a_k^{12}}{\phi(a_k)} \equiv 3\sigma_2^2\sigma_3 - 2\sigma_3\sigma_4.$$

Since $a_k' = 0$ $(k=\overline{1,6})$, the first relation of the system (4) is

$$\sum_{k=1}^{6} B_k = -2D. \tag{23}$$

Hence
$$\sum_{k=1}^{6} B_k = \sigma_2\psi\ \sigma_1 + \psi\sum_{k=1}^{6} a_k^3 - 2D = s_3\psi - 2D. \tag{24}$$

From the equalities (23) and (24) the next theorem follows.

Theorem 1. *System* (4_1)*,* (5) *with respect to constantsa*$_k$ $(k=\overline{1,6})$ *is compatible and it has two solutions:*

1) $\quad B_k = -\frac{1}{3}\ D\ (k=\overline{1,6})\quad$ *(for* $\psi = 0$*),* (25)

or

2) $\quad B_k\ \ (k=\overline{1,6})$ *is determined according to (20) (for* $s_3 = 0$*).*

Then we shall find a solution of the systems (6), (7). The system (6) for our case has the form
$$\left(\frac{2}{a_k} - \sum_j \frac{1}{a_k - a_j}\right)\ C_k + \sum_j \frac{C_j}{a_k - a_j} =$$
$$= (\delta_k\psi - \frac{1}{3}\ D)^2 - \delta_k\psi' + \frac{1}{3}D' + \delta_k\psi D - \frac{1}{3}D^2 - E \quad (k,j=\overline{1,6};\ j\neq k). \tag{26}$$

The determinant of system (26) is equal to zero. By setting
$$C_k = \frac{a_k}{3}\ (E - \frac{1}{3}D' + \frac{2}{9}D^2) + \chi_k \quad (k=\overline{1,6}) \tag{27}$$

in the system (26) we get the system
$$\left(\frac{2}{a_k} - \sum_j \frac{1}{a_k - a_j}\right)\ \chi_k + \sum_j \frac{\chi_j}{a_k - a_j} = (\frac{1}{3}D\psi - \psi')\delta_k + \psi^2\delta_k^2 \quad (k,j=\overline{1,6};\ j\neq k). \tag{28}$$

Applying Cramer's rule to the system (28), we find the value of the determinant
$$\begin{vmatrix}
(\frac{1}{3}D\psi-\psi')\delta_1+\psi^2\delta_1^2 & \frac{1}{a_1-a_2} & \frac{1}{a_1-a_3} & \frac{1}{a_1-a_4} & \frac{1}{a_1-a_5} & \frac{1}{a_1-a_6}\\
(\frac{1}{3}D\psi-\psi')\delta_2+\psi^2\delta_2^2 & a_{22} & \frac{1}{a_2-a_3} & \frac{1}{a_2-a_4} & \frac{1}{a_2-a_5} & \frac{1}{a_2-a_6}\\
(\frac{1}{3}D\psi-\psi')\delta_3+\psi^2\delta_3^2 & \frac{1}{a_3-a_2} & a_{33} & \frac{1}{a_3-a_4} & \frac{1}{a_3-a_5} & \frac{1}{a_3-a_6}\\
(\frac{1}{3}D\psi-\psi')\delta_4+\psi^2\delta_4^2 & \frac{1}{a_4-a_2} & \frac{1}{a_4-a_3} & a_{44} & \frac{1}{a_4-a_5} & \frac{1}{a_4-a_6}\\
(\frac{1}{3}D\psi-\psi')\delta_5+\psi^2\delta_5^2 & \frac{1}{a_5-a_2} & \frac{1}{a_5-a_3} & \frac{1}{a_5-a_4} & a_{55} & \frac{1}{a_5-a_6}\\
(\frac{1}{3}D\psi-\psi')\delta_6+\psi^2\delta_6^2 & \frac{1}{a_6-a_2} & \frac{1}{a_6-a_3} & \frac{1}{a_6-a_4} & \frac{1}{a_6-a_5} & a_{66}
\end{vmatrix},$$

where $a_{ii} = \frac{2}{a_i} - \sum_{j\ (j\neq i)} \frac{1}{a_i - a_j}$ $(i=\overline{2,6})$. A simple calculation shows us that this determinant is equal to
$$\frac{12a_1}{\sigma_6}\ (\sigma_3 - a_1\sigma_2 - a_1^3)\ \left(\frac{1}{3}D\psi - \psi' + \sigma_3\psi^2\right).$$

Also this determinant must be equal to zero because the determinant of the system (28) is equal to zero. Hence we obtain the condition for the function ψ
$$\frac{1}{3}D\psi - \psi' + \sigma_3\psi^2 = 0. \tag{29}$$

Now consider the first case when $\psi = 0$. Then we get

$$B_k = -\frac{1}{3}\ D \quad (k = \overline{1,6}). \tag{30}$$

Solving the system

$$\left(\frac{2}{a_k} - \sum_j \frac{1}{a_k - a_j}\right)\ \chi_k + \sum_j \frac{\chi_j}{a_k - a_j} = 0 \quad (k, j = \overline{1,6};\ j \neq k). \tag{31}$$

with unknown functions $\chi_k \ \ (k = \overline{1,6})$, we find

$$\chi_k = \frac{\xi_i}{\xi_6}\ \chi_6 \quad (i = \overline{1,5}), \tag{32}$$

where

$$\xi_1 = a_1(a_2a_3a_4 + a_2a_3a_5 + a_2a_3a_6 + a_2a_4a_5 + a_2a_4a_6 + a_2a_5a_6 + a_3a_4a_5 +$$

$$a_3a_4a_6 + a_3a_5a_6 + a_4a_5a_6) = a_1(\sigma_3 - a_1\sigma_2 - a_1^3),$$

. . . .

$$\xi_6 = a_6(a_1a_2a_3 + a_1a_2a_4 + a_1a_2a_5 + a_1a_3a_4 + a_1a_3a_5 + a_1a_4a_5 + a_2a_3a_4 +$$

$$a_2a_3a_5 + a_2a_4a_5 + a_3a_4a_5) = a_6(\sigma_3 - a_6\sigma_2 - a_6^3),$$

χ_6 is an arbitrary analytical function of z. Let us set

$$\xi_k = a_k(\sigma_3 - a_k\sigma_2 - a_k^3), \quad \chi = \frac{\chi_k}{\xi_k} \quad (k = \overline{1,6}). \tag{33}$$

Using relations (32), (33), for our case we write a solution of the system (6) in the form

$$C_k = \frac{a_k}{3}\ (E - \frac{1}{3}D' + \frac{2}{9}D^2) + \xi_k\ \chi \quad (k = \overline{1,6}), \tag{34}$$

where ξ_k are determined according to formulas (33), χ is any analytical function.

Let us consider the second case, when $s_3 = \sigma_3 = 0$. Taking into account relation (27), we obtain system (26) as

$$\left(\frac{2}{a_k} - \sum_j \frac{1}{a_k - a_j}\right)\ \chi_k + \sum_j \frac{\chi_j}{a_k - a_j} = \psi^2\delta_k^2 \quad (k, j = \overline{1,6};\ j \neq k), \tag{35}$$

where the function ψ, according to (29), has the form

$$\psi = C\ \exp\left(\frac{1}{3}\int D dx\right), \tag{36}$$

(C is an arbitrary constant). We shall seek functions $\chi_k \quad (k=\overline{1,6})$ in the form

$$\chi_k = \psi^2[a_k^3 \left(\frac{1}{3}a_k^4 + \frac{4}{3}\sigma_2 a_k^2 + \sigma_2^2 + \frac{4}{3}\ \sigma_4\right) + \frac{a_k}{3}\ (6\sigma_2\sigma_4 + 2\sigma_6)] + \tilde{\chi_k} \ \ (k=\overline{1,6}). \quad (37)$$

By substituting (37) into the system (35) we obtain the system (31), where functions $\tilde{\chi_k}$ $(k=\overline{1,6})$ are unknown values. Let us consider the solution (32), (33) of the system (31). Because $\sigma_3 = 0$ then solution of the last system is

$$\tilde{\chi_k} = -a_k^2(\sigma_2 + a_k^2)\chi \ \ (k=\overline{1,6}),$$

where χ is an arbitrary analytical function. Hence solution of system (35) is

$$\chi_k = \psi^2[a_k^3 \left(\frac{1}{3}a_k^4 + \frac{4}{3}\sigma_2 a_k^2 + \sigma_2^2 + \frac{4}{3}\ \sigma_4\right) + \frac{a_k}{3}\ (6\sigma_2\sigma_4 + 2\sigma_6)] - a_k^2(\sigma_2 + a_k^2)\chi \ \ (k=\overline{1,6}). \quad (38)$$

Taking into account the substitution (38), we obtain

$$C_k = \psi^2[a_k^3 \left(\frac{1}{3}a_k^4 + \frac{4}{3}\sigma_2 a_k^2 + \sigma_2^2 + \frac{4}{3}\ \sigma_4\right) +$$
$$+ \frac{a_k}{3}\ (E - \frac{1}{3}D' + \frac{2}{9}D^2 + 6\sigma_2\sigma_4 + 2\sigma_6)] - a_k^2(\sigma_2 + a_k^2)\chi \ \ (k=\overline{1,6}). \quad (39)$$

Thus in the second case we have determined functions C_k $(k=\overline{1,6})$ in the form (39). From Theorem 1 and relations (39) the next theorem follows:

Theorem 2. *System* (6) *with respect to constants* a_k $(k=\overline{1,6})$ *is compatible and it has two solutions:*
1) $C_k = \frac{a_k}{3}(E - \frac{1}{3}D' + \frac{2}{9}D^2) + \xi_k\chi$ *(for* $\psi = 0$*),* (40)
or
2) C_k $(k=\overline{1,6})$ *is determained according to (39)* *(for* $s_3 = 0$*).*

Now we shall seek a solution of the system (7). By our assumptions this system is

$$-B_kC_k + C_k' - DC_k + F_k \prod_j (a_k - a_j) = 0 \quad (k,j=\overline{1,6};\ j\neq k). \quad (41)$$

Using the notations

$$\phi(a_k) \equiv \prod_j (a_k - a_j) = a_k(6a_k^4 + 4\sigma_2 a_k^2 + 2\sigma_4) \ \ (k=\overline{1,6})$$

and (20) in the general case we rewrite system (41) in the form

$$F_k = \frac{(a_k(\sigma_2 + a_k^2)\psi + 2D/3)C_k - C_k'}{\phi(a_k)} \ \ (k=\overline{1,6}).$$

Using relations (39) from the last system we find

$$F_k = [3\psi^3 a_k^{10} + 15\sigma_2\psi^3 a_k^8 + \psi(2D\psi - 6\psi' - 9\chi)a_k^7 + \psi^3(9\lambda_1 + 12\sigma_2^2)a_k^6 +$$

$$+2\sigma_2\psi(4D\psi-9\chi-12\psi')a_k^5+(3\psi^3(\lambda_2+3\lambda_1\sigma_2+\gamma)+9\chi'-6D\chi)a_k^4+$$
$$+3\psi(2D\lambda_1\psi-3\sigma_2^2\chi-6\lambda_1\psi')a_k^3+(3\sigma_2\psi^3(\lambda_2+\gamma)+3\sigma_2(3\chi'-2D\chi))a_k^2+ \qquad (42)$$
$$+\psi(2D\lambda_2\psi+2D\gamma\psi-3\psi\gamma'-6\lambda_2\psi'-6\gamma\psi')a_k] \ \ (k=\overline{1,6}),$$

where $\gamma \equiv E-\frac{1}{3}D'+\frac{2}{9}D^2$, $\lambda_1 \equiv \sigma_2^2+\frac{4}{3}\sigma_4$, $\lambda_2 \equiv \sigma_2\sigma_4+2\sigma_6$. We find the exact form of the functions F_k $(k=\overline{1,6})$ for each of two cases. Consider the case 1 ($\psi=0$, the functions B_k $(k=\overline{1,6})$ are determined according to relations (30), C_k $(k=\overline{1,6})$ - according to (40)). Taking into account relations (42) we obtain functions F_k $(k=\overline{1,6})$ as

$$F_k=\frac{a_k}{3\phi(a_k)}\ \gamma_1+\frac{\xi_k}{\phi(a_k)}\ (\chi'+\frac{2}{3}D\chi) \ \ (k=\overline{1,6}), \qquad (43)$$

where

$$\gamma_1=\frac{1}{3}D''-\frac{2}{3}DD'-E'+\frac{2}{3}DE+\frac{4}{27}D^3. \qquad (44)$$

Since equality (5_2) : $\sum_{k=1}^{6}\ F_k=0$ is true, then $\chi'+\frac{2}{3}D\chi=0$ or

$$\chi=C\ \exp(-\frac{2}{3}\int Ddx), \qquad (45)$$

where C is an arbitrary constant. From relations (43) and (45) we find

$$F_k=\frac{a_k}{3\phi(a_k)}\ \gamma_1 \ \ (k=\overline{1,6}). \qquad (46)$$

Thus in the first case equation (1) has the form

$$w'''=\sum_{k=1}^{6}\frac{w'w''-(a_k)^{-1}\ w'^3+B_kw'^2+C_kw'}{w-a_k}+Dw''+Ew'+\prod_{i=1}^{6}(w-a_i)\ \sum_{k=1}^{6}\frac{F_k}{w-a_k}, \qquad (47)$$

where B_k $(k=\overline{1,6})$ are determined according to formulas (30), C_k $(k=\overline{1,6})$ - according to formulas (40), F_k $(k=\overline{1,6})$ - according to formulas (46) and χ - according to formula (45).

Consider the second case ($s_3=0$ or, taking into account the relation $s_3=3\sigma_3$, we have $\sigma_3=0$). Functions B_k $(k=\overline{1,6})$ are determined from (20), functions C_k $(k=\overline{1,6})$ are determined from (39). Then relations (22) are

$$\sum_{k=1}^{6}\frac{a_k^n}{\phi(a_k)}\equiv 0 \ \ (n=\overline{0,4},6,8,10,12), \ \ \sum_{k=1}^{6}\frac{a_k^5}{\phi(a_k)}\equiv 1, \ \ \sum_{k=1}^{6}\frac{a_k^7}{\phi(a_k)}\equiv -\sigma_2,$$
$$\sum_{k=1}^{6}\frac{a_k^9}{\phi(a_k)}\equiv \sigma_2^2-\sigma_4, \ \ \sum_{k=1}^{6}\frac{a_k^{11}}{\phi(a_k)}\equiv -\sigma_2^3+2\sigma_2\sigma_4-\sigma_6. \qquad (48)$$

To find functions $F_k \ \ (k=\overline{1,6})$ we use the identities (48). Substituting values of the functions $F_k \ \ (k=\overline{1,6})$ from (42) in the relation $\sum_{k=1}^{6} F_k = 0$, we obtain

$$3\sigma_2\psi(2D\psi - 6\psi' - 3\chi) = 0. \tag{49}$$

Since $\psi \neq 0$ (otherwise we have the first case), then from (49) it follows:

$$\sigma_2 = 0, \tag{50}$$

or

$$\chi = \frac{2}{3}D\psi - 2\psi'. \tag{51}$$

Substituting the value of function ψ from (36) into (51), we obtain

$$\chi = 0. \tag{52}$$

Substitute (42), (48) and (52) into the relations $\sum_{k=1}^{6} a_k F_k = 0$, $\sum_{k=1}^{6} a_k^2 F_k = 0$. Then the first relation becomes

$$\frac{1}{3}\ \psi^3\ (2D^2 + 3(3E - 3\sigma_2\sigma_4 + 3\sigma_6 - D') = 0, \tag{53}$$

and the second one be the identity. From (53) we find the value of function E

$$E = \frac{1}{9}(3D' - 2D^2) + \sigma_2\sigma_4 - \sigma_6. \tag{54}$$

By substituting (52), (54) into equation (42), we obtain functions $F_k \ \ (k=\overline{1,6})$ as

$$F_k = \frac{a_k^2(\sigma_2 + a_k^2)(a_k^6 + 4\sigma_2 a_k^4 + a_k^2(3\sigma_2^2 + 4\sigma_4) + 7\sigma_2\sigma_4 + \sigma_6)}{3\phi(a_k)}\ \psi^3 \quad (k=\overline{1,6}). \tag{55}$$

Consider here the subcase $\sigma_2 = 0$. Then from the relation $\sum_{k=1}^{6} a_k F_k = 0$, we find

$$\chi' = \frac{1}{27}(18D\chi + (3D' - 2D^2 - 9E - 9\sigma_6)\psi^3). \tag{56}$$

Substitute (42), (48), (56) and $\sigma_2 = 0$ into the relation $\sum_{k=1}^{6} a_k^2 F_k = 0$, which we can write as

$$9\sigma_4\ \chi\ \psi = 0.$$

From the last equation it follows that $\sigma_4 = 0$ (otherwise we obtain one of two considered above cases: $\psi = 0$ or $\chi = 0$). Then functions $F_k \ \ (k=\overline{1,6})$ have the next form:

$$F_k = \frac{1}{27\phi(a_k)}[a_k\psi(9a_k^9\psi^2 - 27a_k^6\chi + 9a_k^3\sigma_6\psi^2 + \psi(3(D'' - 3E') - 4DD'))]\ (k=\overline{1,6}). \tag{57}$$

Thus, in the second case ($\sigma_3 = 0$) we obtain two equations (47), where $B_k \ \ (k=\overline{1,6})$ are determined by (20), $C_k \ \ (k=\overline{1,6})$ are determined by (39), $F_k \ \ (k=\overline{1,6})$ be (55) or (57). The preceding gives the following two theorems.

Theorem 3. *For constants* a_k $(k = \overline{1,6})$ *the system (7) is compatible and it has three solutions:*
1) F_k $(k = \overline{1,6})$ *are determined by (46) and there are (44), (45) (for* $\psi = 0$*)*
or
2) F_k $(k = \overline{1,6})$ *are determined by (55) and there are (52), (54) (for* $s_3 = 0$*)*
or
3) F_k $(k = \overline{1,6})$ *are determined by (57) and there are (56),* $\sigma_2 = \sigma_4 = 0$ *(for* $s_3 = 0$*).*

Theorem 4. *Differential equations (47), where*
1) B_k $(k = \overline{1,6})$ *are determined by (30),* C_k $(k = \overline{1,6})$ *- by (40),* F_k $(k = \overline{1,6})$ *- by (46),* $\chi = C\ \exp(-\frac{2}{3}\int D dx)$ *(* C *is arbitrary constant)*
or
2) B_k $(k = \overline{1,6})$ *are determined by (20),* C_k $(k = \overline{1,6})$ *- by (39),* F_k $(k = \overline{1,6})$ *- by (55) and there are* $\sigma_3 = 0$*, (52), (54)*
or
3) B_k $(k = \overline{1,6})$ *are determined by (20),* C_k $(k = \overline{1,6})$ *- by (39),* F_k $(k = \overline{1,6})$ *- by (57) and there are* $\sigma_2 = \sigma_3 = \sigma_4 = 0$*, (56) belong to P-type.*

By theorem 4 we obtain three classes of differential equations (47) of P-type.

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