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On initial value problem in theory of the second order differential equations

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Abstract. We consider the properties of the second order nonlinear differential equations $b'' = g(a, b, b')$ with the function $g(a, b, b' = c)$ satisfying the following nonlinear partial differential equation

$$g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2g_{bbcc} + 2cgg_{bccc} + g^2g_{cccc} + (g_a + cg_b)g_{ccc}-$$
$$4g_{abc} - 4cg_{bbc} - cg_cg_{bcc} - 3gg_{bcc} - g_cg_{acc} + 4g_cg_{bc} - 3g_bg_{cc} + 6g_{bb} = 0\,.$$

Any equation $b'' = g(a, b, b')$ with this condition on the function $g(a, b, b')$ has the General Integral $F(a, b, x, y) = 0$ shared with General Integral of the second order ODE's $y'' = f(x, y, y')$ with the condition $\frac{\partial^4 f}{\partial y'^4} = 0$ on the function $f(x, y, y')$ or $y''+a_1(x, y)y'^3+3a_2(x, y)y'^2+3a_3(x, y)y'+a_4(x, y) = 0$ with some coefficients $a_i(x, y)$.

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1 Introduction

The relation between the equations in the form

$$y'' + a_1(x, y)y^2 + 3a_3(x, y)y' + a_4(x, y) = 0 \tag{1}$$

and

$$b'' = g(a, b, b') \tag{2}$$

with the function $g(a, b, b')$ satisfying the p.d.e

$$\begin{gathered} g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2g_{bbcc} + 2cgg_{bccc} + g^2g_{cccc} + (g_a + cg_b)g_{ccc}- \\ 4g_{abc} - 4cg_{bbc} - cg_cg_{bcc} - 3gg_{bcc} - g_cg_{acc} + 4g_cg_{bc} - 3g_bg_{cc} + 6g_{bb} = 0. \end{gathered} \tag{3}$$

from geometrical point of view was studied by E. Cartan [1].

In fact, according to the expressions for curvature of the space of linear elements $(x,\ y,\ y')$ connected with equation (1)

$$\Omega^1_2 = a[\omega^2 \wedge \omega^2_1]\,, \quad \Omega^0_1 = b[\omega^1 \wedge \omega^2]\,, \quad \Omega^0_2 = h[\omega^1 \wedge \omega^2] + k[\omega^2 \wedge \omega^2_1]\,,$$

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where
$$a = -\frac{1}{6}\frac{\partial^4 f}{\partial y'^4}, \quad h = \frac{\partial b}{\partial y'}, \quad k = -\frac{\partial \mu}{\partial y'} - \frac{1}{6}\frac{\partial^2 f}{\partial^2 y'}\frac{\partial^3 f}{\partial^3 y'},$$
and
$$\begin{aligned} 6b &= f_{xxy'y'} + 2y' f_{xyy'y'} + 2f f_{xy'y'y'} + y'^2 f_{yyy'y'} + 2y' f f_{yy'y'y'} \\ &+ f^2 f_{y'y'y'y'} + (f_x + y' f_y) f_{y'y'y'} - 4f_{xyy'} - 4y' f_{yyy'} - y' f_{y'} f_{yy'y'} \\ &- 3f f_{yy'y'} - f_{y'} f_{xy'y'} + 4f_{y'} f_{yy'} - 3f_y f_{y'y'} + 6f_{yy} \end{aligned}$$
two types of equations by a natural way are evolved: the first type from the condition $a = 0$ and the second type from the condition $b = 0$.

The first condition $a = 0$ determines the equation in form (1) and the second condition leads to the equation (2) where the function $g(a, b, b')$ satisfies the above p.d.e. (3).

From the elementary point of view the relation between both equations (1) and (2) is a result of special properties of their General Integral $F(x, y, a, b) = 0$. So we have the following fundamental diagram:

$$\begin{array}{ccccc} & & F(x,y,a,b)=0 & & \\ & \swarrow\nearrow & & \searrow\nwarrow & \\ y''=f(x,y,y') & & & & b''=g(a,b,b') \\ \Updownarrow & & & & \Updownarrow \\ M^3(x,y,y') & & \Longleftrightarrow & & N^3(a,b,b') \end{array}$$

which presents the General Integral $F(x, y, a, b) = 0$ (as some 3-dim orbifold) in the form of the twice nontrivial fibre bundles on circles over corresponding surfaces:
$$M^3(x, y, y') = U^2(x, y) \times S^1 \quad \text{and} \quad N^3(a, b, b') = V^2(a, b) \times S^1 .$$

2 Examples of solutions of dual equation

Let us consider the solutions of equation (3).

It has many types of reductions and the simplest of them are
$$g = c^\alpha \omega[ac^{\alpha-1}], \quad g = c^\alpha \omega[bc^{\alpha-2}], \quad g = c^\alpha \omega[ac^{\alpha-1}, bc^{\alpha-2}],$$
$$g = a^{-\alpha}\omega[ca^{\alpha-1}], \quad g = b^{1-2\alpha}\omega[cb^{\alpha-1}], \quad g = a^{-1}\omega(c - b/a),$$
$$g = a^{-3}\omega[b/a, b - ac], \quad g = a^{\beta/\alpha-2}\omega[b^\alpha/a^\beta, c^\alpha/a^{\beta-\alpha}].$$

For any type of reduction we can write the corresponding equation (2) and then integrate it.

For example, for the function $g = a^{-\gamma}A(ca^{\gamma-1})$ we get the equation
$$[A+(\gamma-1)\xi]^2 A^{IV} + 3(\gamma-2)[A+(\gamma-1)\xi]A^{III} + (2-\gamma)A^I A^{II} + (\gamma^2 - 5\gamma + 6)A^{II} = 0.$$

One solution of this equation is

$$A = (2-\gamma)[\xi(1+\xi^2) + (1+\xi^2)^{3/2}] + (1-\gamma)\xi.$$

This solution corresponds to the equation

$$b'' = \frac{1}{a}[b'(1+b'^2) + (1+b'^2)^{3/2}]$$

with the General Integral

$$F(x,y,a,b) = (y+b)^2 + a^2 - 2ax = 0.$$

The dual equation has the form

$$y'' = -\frac{1}{2x}(y'^3 + y').$$

Remark that the first examples of solutions of equation (3) were obtained in [3-6].

Proposition 1. *The equation (3) can be represented in the form*

$$\begin{gathered} g_{ac} + gg_{cc} - g_c^2/2 + cg_{bc} - 2g_b = h(a,b,c), \\ h_{ac} + gh_{cc} - g_c h_c + ch_{bc} - 3h_b = 0. \end{gathered} \tag{4}$$

From this it follows that there exists the class of equations (2) with the function $g(a,b,c)$ satisfying the condition

$$g_{ac} + gg_{cc} - g_c^2/2 + cg_{bc} - 2g_b = 0 \tag{5}$$

which is easier solved than equation (3).

Here we present some solutions of the equation (5) as functions depending on two variables $g = g(a,c)$

In the case when $g = g(a,c)$ and $h = 0$ we have the equation

$$g_{ac} + gg_{cc} - \frac{1}{2}g_c^2 = 0\,.$$

To integrate this equation we can transform it into a more convenient form using the variable $g_c = f(a,c)$. Then one obtains:

$$2f_c f_{ac} + (f^2 - 2f_a)f_{cc} = 0\,.$$

After the Legendre transformation we obtain the equation:

$$[(\xi\omega_\xi + \eta\omega_\eta - \omega)^2 - 2\xi]\omega_{\xi\xi} - 2\eta\omega_{\xi\eta} = 0\,.$$

Using the new variable $\xi\omega_\xi + \eta\omega_\eta - \omega = R$ we have the new equation for R:

$$R_\xi - \frac{1}{2}R^2\omega_{\xi\xi} = 0$$

and the following relations:

$$\omega_\eta = \frac{\omega}{\eta} + \frac{R}{\eta} + \frac{2\xi}{\eta R} - \frac{\xi A(\eta)}{\eta}\,, \qquad \omega_\xi = -\frac{2}{R} + A(\eta)$$

with an arbitrary function $A(\eta)$. From the conditions of compatibility it follows:

$$2\eta R_\eta + R_\xi(2\xi - R^2) + \eta A_\eta R^2 = 0\,.$$

Integrating this equation we can obtain general integral.

In the particular case $A = \frac{1}{\eta}$ we have:

$$\frac{R^2}{R - 2\eta} = -\frac{\xi}{\eta} + \Phi\Big(\frac{1}{\eta} - \frac{2}{R}\Big)\,.$$

By the condition $A = 0$ we obtain the equation $2\eta R_\eta + (2\xi - R^2)R_\xi = 0\,,$ which has the solution:

$$R^2 = 2\xi + 2\eta\Phi(R)\,,$$

were $\Phi(R)$ is an arbitrary function.

After choosing the function $\Phi(R)$ we can find the function ω and then using the inverse Legendre transformation, the function g which determines dual equation $b'' = g(a, c)$.

Remark 1. *The solutions of the equations of type*

$$u_{xy} = uu_{xx} + \varepsilon u_x^2 \tag{6}$$

were constructed in [7]. In the article [8] it was showed that they can be presented in the form

$$u = B'(y) + \int [A(z) - \varepsilon y]^{(1-\varepsilon)/\varepsilon} dz,$$

$$x = -B(y) + \int [A(z) - \varepsilon y]^{1/\varepsilon} dz.$$

To integrate the above equations we can apply the parametric representation

$$u = A(a) + U(a, \tau), \quad y = B(a) + V(a, \tau). \tag{7}$$

Using the formulas

$$u_y = \frac{u_\tau}{y_\tau}, \quad u_x = u_x + u_\tau \tau_x$$

we get after the substitution in (6) the conditions

$$A(x) = \frac{dB}{dx} \qquad \text{and} \qquad U_{x\tau} - \left(\frac{V_x U_\tau}{V_\tau}\right)_\tau + U\left(\frac{U_\tau}{V_\tau}\right)_\tau - \frac{1}{2}\frac{U_\tau^2}{V_\tau} = 0.$$

So we get one equation for two functions $U(x,\tau)$ and $V(x,\tau)$. Any solution of this equation determines the solution of equation (6).

Let us consider some examples.

$$A = B = 0, \quad U = 2\tau - \frac{x\tau^2}{2}, \quad V = x\tau - 2\ln(\tau).$$

Using the representation $U = \tau\omega_\tau - \omega, \quad V = \omega_\tau$ it is possible to obtain other solutions of this equation.

The equation $g_{ac} = gg_{cc} - g_c^2/2$ can be integrated in explicit form and the solutions are

$$g = -H'(a) + \int \frac{dz}{[A(z) + \frac{1}{2}a]^3}\,, \qquad c = H(a) + \int \frac{dz}{[A(z) + \frac{1}{2}a]^2}\,,$$

with arbitrary functions $H(a)$ and $A(z)$.

In fact, for $A(z) = z$ we have

$$g = -H'(a) + \int \frac{dz}{[z + \frac{1}{2}a]^3} = -H'(a) - \frac{1}{2}\frac{1}{[z + \frac{1}{2}a]^2}$$

and

$$c = H(a) + \int \frac{dz}{[z + \frac{1}{2}a]^2} = H(a) - \frac{1}{[z + \frac{1}{2}a]^3}\,.$$

As result we get the solution.

Remark 2. *In general case the equation* $g_{acc} + gg_{ccc} = 0$ *is equivalent to the equation*

$$g_{ac} + gg_{cc} - \frac{1}{2}{g_c}^2 = B(a)\,.$$

It can be integrated with the help of Legendre transformation as in the previous case.

Really, we get

$$[(\xi\omega_\xi + \eta\omega_\eta - \omega)^2 - 2\xi + 2B(\omega_\xi)]\omega_{\xi\xi} - 2\eta\omega_{\xi\eta} = 0$$

and the relation

$$2R_\xi = [R^2 + 2B(\omega_\xi)\omega_{\xi\xi}.$$

It can be written in the form

$$2\frac{dR}{d\Omega} = R^2 + 2B(\Omega)$$

using the notation $\omega_\xi = \Omega$.

Proposition 2. *In the case* $h \neq 0$ *and* $g = g(a,c)$ *the system (3) is equivalent to the equation*

$$\Theta_a\Big(\frac{\Theta_a}{\Theta_c}\Big)_{ccc} - \Theta_c\Big(\frac{\Theta_a}{\Theta_c}\Big)_{acc} = 1 \tag{8}$$

where

$$g = -\frac{\Theta_a}{\Theta_c}, \quad h_c = \frac{1}{\Theta_c}.$$

To integrate this equation we use the presentation $c = \Omega(\Theta, a)$. From the relations

$$1 = \Omega_\Theta \Theta_c, \quad 0 = \Omega_\Theta \Theta_a + \Omega_c$$

we get

$$\Theta_c = \frac{1}{\Omega_\Theta}, \quad \Theta_a = -\frac{\Omega_a}{\Omega_\Theta} \quad \text{and} \quad \frac{\Omega_a}{\Omega_\Theta}(\Omega_a)_{ccc} + \frac{1}{\Omega_\Theta}(\Omega_a)_{cca} = 1.$$

Now we get

$$\Omega_{ac} = \frac{\Omega_{a\Theta}}{\Omega_\Theta} = (\ln \Omega_\Theta)_a = K, \quad \Omega_{acc} = \frac{K_\Theta}{\Omega_\Theta},$$

$$\Omega_{accc} = (\frac{K_\Theta}{\Omega_\Theta})_\Theta \frac{1}{\Omega_\Theta}, \quad (\Omega_{acc})_a = (\frac{K_\Theta}{\Omega_\Theta})_a - \frac{\Omega_a}{\Omega_\Theta}(\frac{K_\Theta}{\Omega_\Theta})_\Theta.$$

As a result the equation (8) takes the form

$$\left[\frac{(\ln \Omega_\Theta)_{a\Theta}}{\Omega_\Theta}\right]_a = \Omega_\Theta \tag{9}$$

and can be integrated by the substitution $\Omega(\Theta, a) = \Lambda_a$. So, we get the equation

$$\Lambda_{\Theta\Theta} = \frac{1}{6}\Lambda_\Theta^3 + \alpha(\Theta)\Lambda_\Theta^2 + \beta(\Theta)\Lambda(\Theta) + \gamma(\Theta) \tag{10}$$

with arbitrary coefficients α, β, γ.

Let us consider the following examples.

1. $\alpha = \beta = \gamma = 0$

The solution of equation (10) is

$$\Lambda = A(a) - 6\sqrt{B(a) - \frac{1}{3}\Theta}$$

and we get

$$c = A' - \frac{3B'}{\sqrt{B - \frac{1}{3}\Theta}} \qquad \text{or} \qquad \Theta = 3B - 27\frac{B'^2}{(c - A')^2}.$$

This solution corresponds to the equation

$$b'' = -\frac{\Theta_a}{\Theta_c} = -\frac{1}{18B'}b'^3 + \frac{A'}{6B'}b'^2 + \left(\frac{B''}{B'} - \frac{A'^2}{6B'}\right)b' + A'' + \frac{A'^3}{18B'} - \frac{A'B''}{B'}$$

cubical in the first derivative b' with arbitrary coefficients $A(a), B(a)$. This equation is equivalent to the equation $b'' = 0$ under a point transformation.

In fact, from the formulas

$$L_1 = \frac{\partial}{\partial y}(a_{4y} + 3a_2a_4) - \frac{\partial}{\partial x}(2a_{3y} - a_{2x} + a_1a_4) - 3a_3(2a_{3y} - a_{2x}) - a_4a_{1x},$$

$$L_2 = \frac{\partial}{\partial x}(a_{1x} - 3a_1a_3) + \frac{\partial}{\partial x}(a_{3y} - 2a_{2x} + a_1a_4) - 3a_2(a_{3y} - 2a_{2x}) + a_1a_{4y}$$

which determine the components of projective curvature of the space of linear elements for the equation in the form

$$y'' + a_1(x,y)y'^3 + 3a_2(x,y)y'^2 + 3a_3(x,y)y' + a_4(x,y) = 0$$

we have

$$a_1(x,y) = \frac{1}{18B'}, \quad a_2(x,y) = -\frac{A'}{18B'}, \quad a_3(x,y) = \frac{A'^2}{18B'} - \frac{B''}{3B'},$$

$$a_4(x,y) = \frac{A'B''}{B'} - \frac{A'^3}{18B'} - A''$$

and conditions $L_1 = 0, \quad L_2 = 0$ hold.

This means that our equation determines a projective flat structure in the space of elements (x, y, y').

Remark 3. *The conditions* $L_1 = 0, \quad L_2 = 0$ *correspond to the solutions of the equation (3) in the form*

$$g(a,b,b') = A(a,b)b'^3 + 3B(a,b)b'^2 + 3C(a,b)b' + D(a,b).$$

In general case the equation (2) with condition (3) determines the 3-dimensional Einstein-Weyl geometry in the space of linear elements (a, b, b').

For more general classes of the form-invariant equations the notion of dual equation is introduced by analogous way.

For example, for the form-invariant equation of the type

$$P_n(b')b'' - P_{n+3}(b') = 0,$$

where $P_n(b')$ are the polynomials of degree n in b' with coefficients depending on the variables a, b, the dual equation $b'' = g(a,b,b')$ has the right-hand side $g(a,b,b')$ in the form [9]

$$\begin{vmatrix} \psi_{n+4} & \psi_{n+3} & \dots & \psi_4 \\ \psi_{n+5} & \psi_{n+4} & \dots & \psi_5 \\ . & . & \dots & . \\ \psi_{2n+4} & \psi_{2n+3} & \dots & \psi_{n+4} \end{vmatrix} = 0,$$

where the functions ψ_i are determined with the help of the relations

$$4!\psi_4 = -\frac{d^2}{da^2}g_{cc} + 4\frac{d}{da}g_{bc} - g_c(4g_{bc} - \frac{d}{da}g_{cc}) + 3g_bg_{cc} - 6g_{bb},$$

$$i\psi_i = \frac{d}{da}\psi_{i-1} - (i-3)g_c\psi_{i-1} + (i-5)g_b\psi_{i-2}, \quad i > 4.$$

For example, for the equation $2yy'' - y'^4 - y'^2 = 0$ with the solution

$$x = a(t + \sin t) + b, \quad y = a(1 - \cos t)$$

we have the dual equation $b'' = -\tan(b'/2)/a$.

According to the above formulas for $n = 1$ we get

$$4!\psi_4 = \frac{3}{2a^3}\tan\frac{c}{2}(1+\tan^2\frac{c}{2})^3, \qquad 5!\psi_5 = -\frac{15}{4a^4}\tan\frac{c}{2}(1+\tan^2\frac{c}{2})^4,$$

$$6!\psi_6 = \frac{90}{8a^5}\tan\frac{c}{2}(1+\tan^2\frac{c}{2})^5,$$

and the relation

$$\psi_5^2 - \psi_4\psi_6 = 0$$

is satisfied.

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