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The centre-focus problem for analytical systems of Lienard form in degenerate case

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Abstract. For analytical systems of Lienard form in the case of zero eigenvalues of its linear part is obtained the algebraic criterion of the centre existence, which is analogous to the Cherkas's criterion for systems with imaginary eigenvalues of linear part. We give the solution of centre-focus problem for one class of cubic systems.

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1 Introduction

For analytical systems of Lienard form in the case of pure imaginary eigenvalues of linear part L.A.Cherkas gives effective necessary and sufficient conditions of algebraic character for the centre existence [1–3]. For example, for the Lienard system

$$\dot{x} = y,\ \dot{y} = -xf(x) + xg(x)y, \tag{1}$$

where f, g are analytical in the neighborhood of $x = 0$ functions, $f(0) = 1$, he received the following result

Theorem 1. [1] *The origin of coordinate system (1) is a centre if and only if the system of equations*

$$F(x) = F(y),\ G(x) = G(y),$$

where $F(x) = \int_0^x t\,f(t)dt,\ G(x) = \int_0^x t\,g(t)dt$, *has an analytical in the neighborhood of* $x = 0$ *solution* $y = \varphi(x),\ \varphi(0) = 0,\ \varphi'(0) = -1$.

For the systems of type (1), where $f(x) = x^{2n}f_1(x),\ f_1(0) = 1$, the theorem analogous to Theorem 1 was proved in [4, 5].

In the present article we consider the system of differential equations

$$dx/dt = y,\ dy/dt = \sum_{i=0}^{3} p_i(x)y^i, \tag{2}$$

where $p_i(x)$ are analytical in the neighborhood of $x = 0$ functions of the form

$$p_0(x) = -x^{2n-1} + \sum_{k=2n}^{\infty} a_kx^k,\quad p_1(x) = Ax^{n-1} + \sum_{k=n}^{\infty} b_kx^k,$$

$$p_j(x) = \sum_{k=0}^{\infty} \alpha_{k,j} x^k, \ j = 2, 3. \tag{3}$$

If $4n - A^2 > 0$, then the critical point $O(0,0)$ of system (2) is either a centre or a focus [4,6]. We know [4,7] that there exists a formal transformation

$$x = u + \sum_{i+j=2}^{\infty} \alpha_{i,j} u^i v^j, \quad y = v + \sum_{i+j=2}^{\infty} \beta_{i,j} u^i v^j, \tag{4}$$
$$dt = (1 + \sum_{i+j=1}^{\infty} \gamma_{i,j} u^i v^j) d\tau$$

which transforms (1) to a formal system

$$du/d\tau = v + \sum_{k=n} A_k u^k, \ dv/d\tau = -u^{2n-1}, \tag{5}$$

where $A_n = A/n$.

Theorem 2. [7] *The critical point $O(0,0)$ of system (2) is a centre if and only if $A_{2i+1} = 0, \ i = [n/2], [n/2]+1, \ldots,$ in (5).*

Definition 1. *The critical point $O(0,0)$ of system (2), where p_i are analytical functions of (3) type with complex coefficients, is called a centre if there is a formal transformation (4) which transforms (2) to system (5), where $A_{2i+1} = 0, \ i = [n/2], [n/2]+1, \ldots$*

In the present paper we will show the algebraic criterion of the existence of the centre of the system (2) and will give the solution of centre-focus problem for the system

$$\dot{x} = y(1 + Dx + Px^2), \ \dot{y} = -x^3 + Axy + By^2 + Kx^2y + Lxy^2 + My^3, \tag{6}$$

where A, B, C, D, K, L, M are complex constants.

The solution of centre-focus problem for the system (6) where $D = P = 0$ is contained in [4, 8]. There are many works in which the centre-focus problem is solved for various classes of cubic systems in the case of imaginary eigenvalues of linear part (e.g.[9]-[23]).

2 The algebraic criterion for the existence of a centre

Theorem 3. *The critical point $O(0,0)$ of system (2) is a centre if and only if the system of equations*

$$F_1(x) = F_1(y), \ F_2(x) = F_2(y) \tag{7}$$

or the system

$$F_1(x) = F_1(y), \ F_3(x) = F_3(y), \tag{8}$$

where $F_1 = Q_2^3/Q_1^5,\ F_2 = Q_3^3/Q_1^7,\ F_3 = Q_4/Q_1^3,$

$$Q_1 = 2p_1^3 - 9p_0p_1p_2 + 27p_0^2p_3 + 9p_1p_0' - 9p_0p_1',$$
$$Q_2 = Q_1R - p_0Q_1', \quad Q_3 = 5Q_2R - 3p_0Q_2',$$
$$Q_4 = 7Q_3R - 3p_0Q_3', \quad R = p_1^2 - 3p_0p_2 + 3p_0',$$

has a solution $y = \varphi(x)$, *where* $\varphi(x)$ *is an analytical in the neighborhood of* $x = 0$ *function such that* $\varphi(0) = 0,\ \varphi'(0) = -1$ *(we do not exclude the case when one or both equations of systems (7), (8) turn to the identity).*

Proof. Necessity. Suppose that the critical point $O(0,0)$ is a centre for system (2). The change

$$y = z/[v(x)(1+z)], \tag{9}$$

where $v(x)$ is the solution of the differential equation

$$v' = -p_3(x) - p_2(x)v - p_1(x)v^2 - p_0(x)v^3 \tag{10}$$

with the initial condition $v(0) = 1$, and the elimination of the time transform the system (2) to the equation

$$\begin{aligned} v(x)zz' = p_0(x)v^3(x) + [p_1(x)v^2(x) + 3p_0(x)v^3(x)]z+ \\ +[p_1(x)v^2(x) + 2p_0(x)v^3(x) - p_3(x)]z^2. \end{aligned} \tag{11}$$

Then, the change $z = \alpha(x)w$, where $\alpha(x)$ is the solution of the differential equation

$$\alpha'v(x) = \alpha[p_1(x)v^2(x) + 2p_0(x)v^3(x) - p_3(x)], \tag{12}$$

with $\alpha'(0) = 1$, transforms (11) into the equation

$$ww' = f(x) + g(x)w, \tag{13}$$

where $f(x) = p_0(x)[v(x)/\alpha(x)]^2,\ g(x) = [p_1(x) + 3p_0(x)v(x)]v(x)/\alpha(x)$. From the theorem 19.7 from [4] we conclude that $O(0,0)$ of the equation (13) is a centre if and only if the system of equations

$$F(x) = F(y),\ G(x) = G(y),$$

where $F(x) = \int_0^x f(t)dt,\ G(x) = \int_0^x g(t)dt$, has an analytical in the neighbourhood of $x = 0$ solution $y = \varphi(x),\ \varphi(0) = 0,\ \varphi'(0) = -1$. Thus, in the examined case we have

$$F(x) = F[\varphi(x)],\ G(x) = G[\varphi(x)]. \tag{14}$$

From (14) it follows that

$$f(x) = f[\varphi(x)]\varphi'(x),\ g(x) = g[\varphi(x)]\varphi'(x). \tag{15}$$

From (15) we get that $\omega_0(x) = \omega_0[\varphi(x)]$, where

$$\omega_0(x) = f(x)/g(x) = p_0(x)v(x)/[\alpha(x)(p_1(x) + 3p_0(x)v(x))].$$

The differentiation of $\omega_0(x)$ taking into account (10), (12) gives

$$\omega_0'(x)/g(x) + 2/9 = Q_1(x)/[9(p_1(x) + 3p_0(x)v(x))^3]. \tag{16}$$

From (16) we have $\omega_1(x) = \omega_1[\phi(x)]$, where

$$\omega_1(x) = Q_1(x)/[p_1(x) + 3p_0(x)v(x)]^3. \tag{17}$$

The derivation of (17) gives us

$$\omega_1'(x)\omega_0(x)/g(x) - \omega_1^2(x)/3 - \omega_1(x)/3 = -Q_2(x)/[p_1(x) + 3p_0(x)v(x)]^5.$$

Consequently, $\omega_2(x) = \omega_2[\varphi(x)]$, where

$$\omega_2(x) = Q_2(x)/[p_1(x) + 3p_0(x)v(x)]^5. \tag{18}$$

Then (18) gives

$$\omega_2'(x)\omega_0(x)/g(x) - 5\omega_1(x)\omega_2(x)/9 - 5\omega_2(x)/9 = -Q_3(x)/[3(p_1(x) + 3p_0(x)v(x))^7].$$

Thus, $\omega_3(x) = \omega_3[\varphi(x)]$, where

$$\omega_3(x) = Q_3(x)/[p_1(x) + 3p_0(x)v(x)]^7. \tag{19}$$

The derivation of (19) gives

$$\omega_3'(x)\omega_0(x)/g(x) - 7\omega_1(x)\omega_3(x)/9 - 7\omega_3(x)/9 = -Q_4(x)/[3(p_1(x) + 3p_0(x)v(x))^9].$$

Hence, $\omega_4(x) = \omega_4[\varphi(x)]$, where

$$\omega_4(x) = Q_4(x)/[p_1(x) + 3p_0(x)v(x)]^9. \tag{20}$$

From (17), (18) we have $F_1(x) = F_1[\varphi(x)]$, from (17), (19) we have $F_2(x) = F_2[\varphi(x)]$, and from (17), (20) we have $F_3(x) = F_3[\varphi(x)]$. The necessity is proved. The sufficiency is proved in the same way [2].

For system (2), where $p_3(x) = 0$, we have the following result.

Theorem 4. *The critical point $O(0,0)$ of system (2) in the case of $p_3(x) = 0$ is a centre if and only if the system of equations*

$$W_1(x) = W_1(y),\ W_2(x) = W_2(y), \tag{21}$$

where $W_1 = (p_0p_1p_2 - p_1p_0' + p_0p_1')/p_1^3,\ W_2 = W_1'p_0/p_1^2$, has a solution $y = \varphi(x)$, where $\varphi(x)$ is an analytical in the neighbourhood of $x = 0$ function, $\varphi(0) = 0$, $\varphi'(0) = -1$ (we do not exclude the case when one or both equations of system (21) turn into the identities).

3 The solution of centre-focus problem for system (6)

Together with system (6) we will examine the equation

$$yy' = \sum_{i=0}^{3} p_i(x)y^i, \tag{22}$$

where

$$p_0(x) = -x^3/(1 + Dx + Px^2), \qquad p_1(x) = (Ax + Kx^2)/(1 + Dx + Px^2),$$
$$p_2(x) = (B + Lx)/(1 + Dx + Px^2), \quad p_3(x) = M/(1 + Dx + Px^2).$$

By the method [4, 7] we find a formal change for system (6)

$$x = u + \sum_{i+j=2}^{\infty} \alpha_{i,j}u^i v^j, \quad y = v + \sum_{i+j=2}^{\infty} \beta_{i,j}u^i v^j, \tag{23}$$

which transforms (6) to the system

$$du/dt = v + \sum_{i=2}^{\infty} d_i u^i, \quad dv/dt = -u^3 + \sum_{i=4}^{\infty} h_i u^i. \tag{24}$$

If in (23) $\alpha_{0,j} = \beta_{0,j} = 0,\ j = 2, 3, \dots$, then all $d_i,\ h_i$ in (24) are defined uniquely. In this case in (22)

$$\begin{aligned}
&d_2 = A/2, \qquad d_3 = A(B + D)/6 + K/3,\\
&d_4 = A(B + D)(2B + D)/24 + K(B + D)/4 + A(L + 2P)/24,\\
&d_5 = A(B + D)(2B + D)(3B + D)/120 + K(B + D)(11B + 7D)/60+\\
&\qquad AL(7B + 5D)/120 - M(A^2 - 18)/30 + AP(3B + 2D)/30 + K(L + 2P)/15;\\
&h_4 = -(B + 3D)/2, \quad h_5 = -(B + D)(B + 5D)/4 - P,\\
&h_6 = -(B + D)(B^2 + 9BD + 6D^2)/8 + L(B - 5D)/24 - AM/6 - P(3B + 4D)/2;
\end{aligned}$$

$d_i,\ i = \overline{6, 15}$, are polynomials of A, B, D, K, L, M, P, which consist accordingly of 27, 47, 75, 117, 172, 251, 350, 485, 651, 869 addends; $h_i,\ i = \overline{7, 16}$, are polynomials, which consist accordingly of 17, 27, 45, 67, 102, 145, 208, 284, 391, 518 addends.

The change of $u_1 = \varphi(u) = u(1 - \sum_{k=4}^{\infty} h_k u^{k-3})^{1/4},\ d\tau = (1 - \sum_{k=4}^{\infty} h_k u^{k-3})dt$ reduces system (24) to the form

$$du_1/d\tau = v + \sum_{k=2}^{\infty} d_k[\varphi^{-1}(u_1)]^k = v + \sum_{k=2}^{\infty} A_k u_1^k,\ dv/d\tau = -u_1^3. \tag{25}$$

The values $f_i = A_{2i+1},\ i = 1, 2, \dots$, where A_{2i+1} is from (25) will be called the focus values of system (6). Focus values $f_k,\ k = 1, 2, \dots$, are the polynomials from the ring $\mathbb{C}[K, M, L, P, D, B, A]$; $f_i,\ i = \overline{1, 7}$, contain accordingly 3, 15, 47, 117, 251, 485, 869 addends.

Let's generate the ideal [24] $I = \langle f_1, f_2, ..., f_k, ...\rangle \subset \mathbb{C}[K, M, L, P, D, B, A]$. Let us denote by $\mathbb{V}(I)$ the variety of ideal I [24], i.e. $\mathbb{V}(I) = \{a = (K, M, L, P, D, B, A) \in \mathbb{C}^7 : \text{for any} f \in I,\ f(a) = 0\}$.

Definition 2. *The set $W = \mathbb{V}(I)$ is called the variety of the centre of system (6).*

It is obvious that $O(0,0)$ of system (6) is a centre if and only if $a \in W$.

The focus values $f_k,\ k = \overline{1,7}$, can be found with the help of computer system Mathematica 4.1. Instead of focus values $f_k,\ k = \overline{1,7}$, we will examine

$$g_1 = 15f_1 = A(B-2D) + 5K,\ g_k = f_k(mod\langle g_1, ..., g_{k-1}\rangle),\ k = \overline{2,7}.$$

The values g_k can be found with the help of the division algorithm [24]. We have

$$g_2 = A(B-2D)(B^2 - 9BD + 4D^2) + 10AL(3B - D) - 25M(2A^2 - 21) - 5AP(13B - 6D),$$

$$g_3 = A(B-2D)^2(2B+D)(19B^2 + 389BD - 204D^2) - 1250AL^2(3B-D) - 125AL(B-2D)(17B^2 - 3BD - 2D^2) + 5625M(-7B^2 - 2BD + 12D^2 + 20L - 45P) + 125AP(B-2D)(53B^2 + 16BD - 24D^2) + 625AP[P(29B - 18D) - L(B - 2D)].$$

Let us note that $g_k,\ k = \overline{4,7}$, contains accordingly 51, 90, 143, 211 addends. In so doing $I = \langle f_1, ..., f_k, ...\rangle = \langle g_1, ..., g_k, ...\rangle$. We put $I_k = \langle f_1, ..., f_k\rangle$. Then $I_k = \langle g_1, ..., g_k\rangle$.

Theorem 5. *The variety of the centre of system (6) can be represented in the form $W = \mathbb{V}(J_1) \bigcup \mathbb{V}(J_2) \bigcup ... \bigcup \mathbb{V}(J_{14})$, where*

$J_1 = \langle A,\ M,\ K\rangle,\quad J_2 = \langle B,\ D,\ M,\ K\rangle,\quad J_3 = \langle B - 2D,\ L - 2P,\ M,\ K\rangle,$

$J_4 = \langle 3B - D,\ P - 2B^2,\ M,\ AB - K\rangle,$

$J_5 = \langle (B - 2D)(B + 3D) + 25P,\ (B - 2D)(3B - D) + 25L,\ M,\ A(B - 2D) + 5K\rangle,$

$J_6 = \langle 2B - D, 9B^2 - 25P, 3B^2 + 25L, M, 3AB - 5K\rangle,$

$J_7 = \langle 17B - 4D,\ 9B^2 - 2P,\ 3B^2 - 2L,\ M,\ 3AB - 2K\rangle,$

$J_8 = \langle 7B - 4D,\ B^2 + 2P,\ B^2 + L,\ M,\ AB - 2K\rangle,$

$J_9 = \langle A^2 - 6,\ 3(B - 2D)(3B + 4D) + 100P,\ A(17B - 4D)(B - 2D)^2 + 4500M,\ B(B - 2D) + 5L,\ A(B - 2D) + 5K\rangle,$

$J_{10} = \langle A^2 - 6,\ 3(B - 2D)(3B - D) + 25P,\ (17B - 9D)(B - 2D) + 25L,\ 2A(B - 2D)^2(2B - D) + 225M,\ A(B - 2D) + 5K\rangle,$

$J_{11} = \langle A^2 - 6,\ (3B - D)(3B + 4D) + 25P,\ 11B^2 + BD + 4D^2 + 25L,\ 2A(7B - 4D)(2B + D)^2 + 1125M,\ A(B - 2D) + 5K\rangle,$

$J_{12} = \langle A^2 - 6,\ 3B - D,\ 3(2B^2 + L) - 4P, AB(2B^2 - P) - 9M,\ AB - K\rangle,$

$J_{13} = \langle A - 3, (B - 7D)(B - 2D) + 25(L - 2P), 3(B - 2D) + 5K, -(B - 2D)^2(B + 3D) - 25P(B - 2D) + 125M\rangle,$

$J_{14} = \langle A + 3,\ (B - 7D)(B - 2D) + 25(L - 2P),\ -3(B - 2D) + 5K,\ (B - 2D)^2(B + 3D) + 25P(B - 2D) + 125M\rangle$

and $\mathbb{V}(J_i),\ i = \overline{1,14}$, are irreducible.

Proposition 1. *If $2A^2 - 7 = 0,\ 3B - D = 0,\ 3B^2 - P = 0,\ 2B^2 - L = 0,\ AB^3 + 14M = 0,\ AB - K = 0,\ B \neq 0$, then $O(0,0)$ of system (6) is a focus of 8^{th} order.*

Proof. In the examined case the system (6) looks as

$$\dot{x} = y(1 + 3Bx + 3B^2x^2),\ \dot{y} = -x^3 + Axy + By^2 + By(Ax^2 + 2Bxy - AB^2y^2/14),\quad (26)$$

where $A^2 = 7/2,\ B \neq 0$. There exists a change (4) which reduces (26) to the system

$$\frac{du}{d\tau} = v + A(u^2/2 - 55B^6u^8/25088 - 5445B^{12}u^{14}/314703872 - 55B^{15}u^{17}/161308784 - 28655B^{18}u^{20}/281974669312 - 3267B^{21}u^{23}/374283822640 + ...), \quad \frac{dv}{d\tau} = -u^3.$$

Hence, the focus values $f_k = 0,\ k = \overline{1,7}$, but $f_8 \neq 0$, i.e. $O(0,0)$ is a focus of 8^{th} order of system (26).

Lemma 1. *Consider $M = 0$. Then the variety of the centre of the system (6) is shown as $V_1 = \mathbb{V}(J_1) \bigcup \mathbb{V}(J_2) \bigcup ... \bigcup \mathbb{V}(J_8)$.*

Proof. Let us make the ideal $J_0 = I_7 + \langle M \rangle$. We compute the Groebner basis of J_0 with lex-ordering with the order $K > M > L > P > D > B > A$ and get

$$J_0 = \langle A(7B-4D)(17B-4D)(B-2D)(2B-D)(3B-D)^2[(B-2D)(B+3D)+25P],\ -A(B-2D)[(B-2D)(B+3D)+25P](4427B^4-5798B^3D+2805B^2D^2-608BD^3+48D^4+125B^2P),\ A(B-2D)\,[\,(B-2D)(B+3D)+25P]\,[2(157B^3-157B^2D+69BD^2-16D^3)-25P(4B-3D)],\ A[(B-2D)(B^2-9BD+4D^2)+10L(3B-D)-5P(13B-6D)],\ A\,[2\,(B-2D)(156B^4-1823B^3D+569B^2D^2-142BD^3+96D^4)+6250\,BL\,(2B^2-P)-125\,P\,(239B^3-101B^2D+56BD^2-20D^3)+625P^2(23B-6D)],\ M,\ A(B-2D)+5K\rangle.$$

Hence, $\mathbb{V}(J_0) = \mathbb{V}(J_1) \bigcup \mathbb{V}(J_2) \bigcup ... \bigcup \mathbb{V}(J_8)$. Let us show then that on the set $\mathbb{V}(J_0)$ the equation (22) and so the system (6) have a centre in $O(0,0)$. Indeed, on the sets $\mathbb{V}(J_1),\ \mathbb{V}(J_2)$ we find the cases of symmetry and therefore the equation (22) has a centre in $O(0,0)$. To prove the existence of the centre on the sets $\mathbb{V}(J_k),\ k = \overline{3,8}$, we will use Theorem 4. On the set $\mathbb{V}(J_4)$ for equation (22) the functions $W_1,\ W_2$ from (21) look like

$$W_1(x) = 2/A^2 - Lu(x)/A^2, \quad W_2(x) = -2B^2Lu^2(x)/A^4 + 2Lu(x)/A^4,$$

where $u(x) = x^2/(1+Bx)^2$. Consequently, the equation (22) in this case has a centre in $O(0.0)$. On the sets $\mathbb{V}(J_3),\ \mathbb{V}(J_5)$ the existence of the centre follows from the fact that $W_1 = 2/A^2$. On the set $\mathbb{V}(J_6)$

$$W_1(x) = 2/A^2 - 18B^2u(x)/A^2, \quad W_2(x) = 36B^2u(x)/A^4 - 972B^4u^2(x)/A^4,$$

where $u(x) = x^2(5+Bx)/(5+3Bx)^3$; $O(0,0)$ of the equation (22) is a centre. On the set $\mathbb{V}(J_7)$ the equation (22) has a centre in $O(0,0)$ because

$$W_1(x) = 2/A^2 - 9B^2u(x)/A^2, \quad W_2(x) = 18B^2u(x)/A^4 - 243B^4u^2(x)/A^4,$$

where $u(x) = x^2(1+2Bx)/(2+3Bx)^3$. On the variety $\mathbb{V}(J_8)$

$$W_1(x) = 2/A^2 - 9B^2u(x)/A^2, \quad W_2(x) = 18B^2u(x)/A^4 - 243B^4u^2(x)/A^4,$$

where $u(x) = x^2/(2+Bx)^3$; $O(0,0)$ of the equation (22) is a centre.

Lemma 2. *Consider $A^2-6=0$. Then the variety of the centre of system (6) can be shown in the following way:*

$$V_2=\mathbb{V}(J_2+\langle A^2-6\rangle)\bigcup\mathbb{V}(J_3+\langle A^2-6\rangle)\bigcup\mathbb{V}(J_4+\langle A^2-6\rangle)\bigcup \\ \mathbb{V}(J_5+\langle A^2-6\rangle)\bigcup\mathbb{V}(J_9)\bigcup\mathbb{V}(J_{10})\bigcup\mathbb{V}(J_{11})\bigcup\mathbb{V}(J_{12}).$$

Proof. When we compute the Groebner basis of the ideal $S=I_7+\langle A^2-6\rangle$ we have $S=\langle h_1,...,h_{27}\rangle$, where

$h_1=A^2-6,\dots,h_4=(B-2D)(3B-D)^4[(B-2D)(B+3D)+25P][(3B-D)\times (3B+4D)+25P][3(B-2D)(3B-D)+25P][3(B-2D)(3B+4D)+100P],\dots,$
$h_7=-B^5(B-2D)(3B-D)[(B-2D)(3B-D)(1701B^6-78732B^5D+538335B^4D^2- 584060B^3D^3-7860B^2D^4+285408BD^5-69696D^6)-5000BDL(9B-8D)\times (4B-3D)(3B-D)(9B+2D)+625P(3B-D)(135B^5+4617B^4D-6230B^3D^2- 2508B^2D^3+7848\,BD^4-2016\,D^5)+625P^2(B-2D)\,(6489B^3+22071B^2D- 14852BD^2+8D^3)+390625P^3(63B^2-6BD-56D^2+108P)],\dots,$
$h_{24}=-(B-2D)(213B^4-804B^3D-1663B^2D^2+2734BD^3-792D^4)-125L(21B^3+ 79B^2D-168BD^2+52D^3)+6250L^2(3B-D)+125P(29B^3+173B^2D- 310BD^2+96D^3)-3125LP(21B-8D)+2500P^2(22B-9D),\dots,$
$h_{26}=A(B-2D)(B^2-9BD+4D^2)+10AL(3B-D)+225M-5AP(13B-6D),$
$h_{27}=A(B-2D)+5K.$

Hence, $\mathbb{V}(S)=V_2$. From Lemma 1 it follows that on the set $\mathbb{V}(J_k+\langle A^2-6\rangle)$, $k=\overline{2,5}$, $O(0,0)$ of the equation (22) is a centre. On the sets $\mathbb{V}(J_k)$, $k=\overline{9,12}$, the presence in $O(0,0)$ of the centre of the equation (22) follows from the fact that here $F_2=0$, where F_2 is from (7).

Remark 1. *On the set $\mathbb{V}(J_9)$ the system (6) has the integrating factor of Darboux form* $R_1(x,y)=[1-3(B-2D)x/10]^{1/3}[1+(3B+4D)x/10]^{-1}/[x^4-Ax^2y+2y^2+ (B-2D)(Ax^2-4y)xy/5+2A(B-2D)^3xy^3/1125-(B-2D)^2(Ay-12x^2)y^2/150]$, *since* $\dfrac{\partial}{\partial x}\left[y(1+Dx+Px^2)R_1(x,y)\right]+\dfrac{\partial}{\partial y}\left[(-x^3+Axy+By^2+Kx^2y+ Lxy^2+My^3)R_1(x,y)\right]=0$. *On the sets $\mathbb{V}(J_{10})$, $\mathbb{V}(J_{11})$, $\mathbb{V}(J_{12})$ the integrating factors of the system (6) are, accordingly, the functions $R_2(x,y)$, $R_3(x,y)$, $R_4(x,y)$, where*

$R_2(x,y)=[1-3(B-2D)x/5]^{-1/3}[1+(3B-D)x/5]^{-1}/[x^4-Ax^2y+2y^2+(B-2D)\times (Ax^2-4y)xy/5+2A(B-2D)^3xy^3/1125-2(B-2D)^2(Ay-3x^2)y^2/75],$
$R_3(x,y)=[1-(3B-D)x/5]^{1/3}[1+(3B+4D)x/5]^{-1/3}/[x^4-Ax^2y+2y^2+(B-2D)\times (Ax^2-4y)xy/5+2(B-2D)^2x^2y^2/25-2A(2B+D)^2y^3/75+2A(7B-4D)\times (2B+D)^2xy^3/1125],$
$R_4(x,y)=[1+3Bx+3(2B^2+L)x^2/4]^{-1/3}/[x^4-Ax^2y+2y^2-Bxy\,(Ax^2-4y)+ A(3L-4B^3x)y^3/18-B^2(Ay-6x^2)y^2/3].$

Remark 2. *On the sets $\mathbb{V}(J_k)$, $k=\overline{9,12}$, the change (9), where $v(x)\,(v(x)\neq 0)$ is, accordingly, the function of the type*

$v(x) = A[1 - (B - 2D)x/5 - (1 - 3(B - 2D)x/10)^{2/3}]/(3x^2)$,
$v(x) = A[1 - (B - 2D)x/5 - (1 - 3(B - 2D)x/5)^{1/3}]/(3x^2)$,
$v(x) = A[1 - (B - 2D)x/5 - (1 - (3B - D)x/5)^{2/3}(1 + (3B + 4D)x/5)^{1/3}]/(3x^2)$,
$v(x) = A[1 + Bx - (1 + 3x(4B + (2B^2 + L)x)/4)^{1/3}]/(3x^2)$,
transforms the equation (22) to equation (11).

Lemma 3. *Consider $A^2 - 9 = 0$. Then the variety of the centre of system (6) is shown as $V_3 = \mathbb{V}(J_{13})\bigcup\mathbb{V}(J_{14})$.*

Proof. The ideal $S_0 = I_7 + \langle A - 3\rangle$ is represented through Groebner basis in the following way: $S_0 = \langle q_1, ..., q_{25}\rangle$, where

$$q_1 = A - 3, \quad q_2 = -B^7(7B - 4D)(17B - 4D)(2B - D)(3B - D)^3[(B - 7D)\times (B - 2D) + 25(L - 2P)], \dots,$$
$$q_{22} = [(B - 7D)(B - 2D) + 25(L - 2P)][563B^3 + 87B^2D - 1154BD^2 + 456D^3 - 650L(3B - D) + 25P(161B - 72D)],$$
$$q_{23} = [(B - 7D)(B - 2D) + 25(L - 2P)][2(14661862B^5 - 23476145B^4D + 12603805B^3D^2 - 2621310B^2D^3 - 155240BD^4 + 154016D^5) - 1543750BL(2B^2 - P) + 125P(160257B^3 - 135683B^2D + 65508BD^2 - 17240D^3) - 625P^2(8969B - 3948D)],$$
$$q_{24} = A(B - 2D)(B^2 - 9BD + 4D^2) + 10L(3B - D) + 25M - 5P(13B - 6D),$$
$$q_{25} = 3(B - 2D) + 5K.$$

In this case $\mathbb{V}(S_0) = \mathbb{V}(J_{13})$. On the set $\mathbb{V}(J_{13})$ the equation (22) has a centre in $O(0,0)$ because here $Q_1 = 0$, and therefore, systems (7), (8) turn into the identities. The case $\mathbb{V}(J_{14})$ is examined in the same way.

Remark 3. *On the set $\mathbb{V}(J_{13})$ the system (6) has the integrating factor of Darboux form $R_5(x,y) = f_2^{S_2}f_3^{S_3}/f_1^3$, where*

$$f_1 = x^2 - [1 - (B - 2D)x/5]y, \quad f_2 = 1 + (D + g)x/2,$$
$$f_3 = 1 + (D - g)x/2, \quad g^2 = D^2 - 4P,$$
$$S_2 = (2B + D)[(2D - B)(D - g) - Pg(3D - 4B)/(D^2 - 4P)]/(25DP),$$
$$S_3 = (2B + D)[(2D - B)(D + g) + Pg(3D - 4B)/(D^2 - 4P)]/(25DP).$$

On the set $\mathbb{V}(J_{14})$ the system (6) has the integrating factor $R_6(x,y) = f_2^{S_2}f_3^{S_3}/f_0^3$, where $f_0 = x^2 + [1 - (B - 2D)x/5]y$.

Lemma 4. *If*

$$M(A^2 - 6)(A^2 - 9) \neq 0, \tag{27}$$

then $O(0,0)$ of system (6) is a focus.

Proof. Finding Groebner basis of the ideal $I_7 + \langle A\rangle$ we have $I_7 + \langle A\rangle = \langle A, M, K\rangle$, i.e. when (27) holds in the case $A = 0$, $O(0,0)$ of system (6) is a focus. The ideal $I_7 + \langle B\rangle$ via Groebner basis looks as

$$I_7 + \langle B\rangle = \langle B, -A(A^2 - 6)(A^2 - 9)D^{11}(6D^2 - 25P), \dots, 5K - 2AD\rangle.$$

After that we have

$$I_7 + \langle B, D(6D^2 - 25P)\rangle = \langle B, D(6D^2 - 25P), -AD^7(A^2 - 6)(2D^2 + 25L),$$
$$64AD^7(2D^2 + 25L) - 474609375M^3, \dots, 2AD + 5K\rangle.$$

Consequently, when $B = 0$ and (27) holds, system (6) has a focus in $O(0,0)$. We find the ideal $I_7 + \langle 3B - D\rangle$ in the form

$$I_7 + \langle 3B - D\rangle = \langle 3B - D, AB^{13}(A^2 - 6)(A^2 - 9)(2A^2 - 7)(P - 2B^2), \dots, AB - K\rangle.$$

Moreover,

$$I_7 + \langle 3B - D, B(2B^2 - P)\rangle = \langle 3B - D, B(2B^2 - P), M^3, \dots, AB - K\rangle$$

and

$$I_7 + \langle 3B - D, 2A^2 - 7\rangle = \langle 2A^2 - 7, 3B - D, B^7(2B^2 - P)(3B^2 - P), AB - K,$$
$$-B(2B^2 - P)(3B^2 - P)^2, -B(2B^2 - P)(B^2 - 2L + P), AB(2B^2 - P) - 14M\rangle.$$

Hence, taking into account Proposition 1, we conclude that when $3B - D = 0$ and (27), $O(0,0)$ of system (6) is a focus. Since

$$I_7 + \langle B - 2D\rangle = \langle B - 2D, -AB^9(A^2 - 6)(L - 2P), \dots, K\rangle,$$
$$I_7 + \langle B - 2D, B(L - 2P)\rangle = \langle B - 2D, B(L - 2P), B^8M, M(13B^6 - 800M^2), \dots, K\rangle,$$

then when $B - 2D = 0$ together with the condition (27) the system (6) also has a focus in $O(0,0)$. For the ideal $I_7 + \langle 4B - 3D\rangle$ the Groebner basis gives

$$I_7 + \langle 4B - 3D\rangle = \langle 4B - 3D, A(A^2 - 6)(A^2 - 9)B^{11}(3P - B^2), \dots, AB - 3K\rangle.$$

In this case

$$I_7 + \langle 4B - 3D, B(B^2 - 3P)\rangle = \langle 4B - 3D, B(B^2 - 3P),$$
$$-A(A^2 - 6)B^7(B^2 - 9L), 64AB^7(B^2 - 9L) - 4782969M^3, \dots, AB - 3K\rangle,$$

i.e. when $4B - 3D = 0$ and (27) holds, $O(0,0)$ of system (6) is a focus. While examining the ideal $I_7 + \langle 2B + D\rangle$, we have

$$I_7 + \langle 2B + D\rangle = \langle 2B + D, AB^9(A^2 - 6)(A^2 - 9)(B^2 - P)^2, \dots, K + AB\rangle.$$

Here

$$I_7 + \langle 2B + D, B(B^2 - P)\rangle = \langle 2B + D, B(B^2 - P), M^3, \dots, AB + K\rangle.$$

Consequently, when $2B + D = 0$, $O(0,0)$ of system (6) is a focus. For the ideal $I_7 + \langle 2B - D\rangle$ we find a representation in the form

$$I_7 + \langle 2B - D\rangle = \langle 2B - D, AB^9(A^2 - 6)(A^2 - 9)(9B^2 - 25P)(21B^2 - 25P), \dots, 3AB - 5K\rangle.$$

In this case $I_7 + \langle 2B - D, B(9B^2 - 25P)(21B^2 - 25P)\rangle = \langle 2B - D, B(9B^2 - 25P)(21B^2 - 25P), -AB^7(A^2 - 9)(3B^2 + 10L - 5P), -432AB^7(3B^2 + 10L - 5P) - 390625M^3, \dots, 3AB - 5K\rangle$. So, in the case when relations (27) and $2B - D = 0$ are fulfilled, $O(0,0)$ of system (6) is a focus. So, when (27) holds and $AB(3B - D)(B - 2D)(4B - 3D)(2B + D)(2B - D) = 0$, $O(0,0)$ is a focus. We will assume that the condition

$$AB(3B - D)(B - 2D)(4B - 3D)(2B + D)(2B - D) \neq 0$$

holds. Applying the change $x = x_1/B,\ y = y_1/B^2,\ dt = Bd\tau$ we will transform system (6) to the form

$$\begin{aligned} dx_1/d\tau &= y_1(1 + Dx_1/B + Px_1^2/B^2),\\ dy_1/d\tau &= -x_1^3 + Ax_1y_1 + y_1^2 + Kx_1^2y_1/B + Lx_1y_1^2/B^2 + My_1^3/B^3. \end{aligned} \tag{28}$$

System (28) shows that it is enough to study the system (6) when $B = 1$, and then to change D, P, K, L, M for $D/B, P/B^2, K/B, L/B^2, M/B^3$. Let us show that when $B = 1$ $A(A^2-6)(A^2-9)(D-3)(2D-1)(3D-4)(D+2)(D-2) \neq 0$, $O(0,0)$ of system (6) is a focus.

Suppose the contrary, that $O(0,0)$ of system (6) is a centre. From $g_1 = 0$ we find $K = A(2D-1)/5$. Taking into account $A(D-3) \neq 0$ we get from $g_2 = 0$ $L = [-A(2D-1)(4D^2-9D+1) - 25M(2A^2-21) + 5AP(6D-13)]/[10A(D-3)]$. Considering L we have $g_i = \alpha_i h_i/[A(D-3)]^{i-2},\ i = \overline{3,7}$, where $\alpha_i \neq 0$,

$$\begin{aligned} h_3 = {} & 4A^2(2D-1)^2(3D+1)(16D^3-69D^2+157D-157) + 625AM[10A^2(2D-1)\times\\ & (2D^2-3D-3) - 3(164D^3-316D^2+33D-33)] + 15625M^2(2A^2-21)\times\\ & (2A^2-57) - 250A^2P(2D-1)(10D^3-33D^2+63D-62) - 3125AMP\times\\ & (28A^2D-54A^2-348D+549) + 1250A^2P^2(2D-1)(3D-4), \end{aligned}$$

$h_i,\ i = \overline{4,7}$, are polynomials in A, D, P, M. Taking into account $h_3 = 0,\ h_i,\ i = \overline{4,7}$, we show that $h_i = \beta_i v_i/[(2D-1)(3D-4)]^{i-3}$, where $p_i \neq 0,\ v_i,\ i = \overline{4,7}$, are polynomials in A, D, P, M of the first degree with respect to P. We shall denote by $R_x(u, v)$ the resultant of polynomials u, v with respect to x. We have

$$\begin{aligned} R_p(v_4, h_3) &= \gamma_4 A^2(A^2-9)(D-3)^2(2D-1)(3D-4)Mr_4,\\ R_p(v_4, v_i) &= \gamma_i A^2(A^2-9)(D-3)^2(2D-1)(3D-4)Mr_i,\ i = \overline{5,7}, \end{aligned}$$

where $\gamma_i \neq 0,\ r_i,\ i = \overline{4,7}$, are polynomials in A, D, M with integer coefficients. As far as $A(A^2-9)(D-2)(2D-1)(3D-4)M \neq 0$, then $r_i = 0,\ i = \overline{4,7}$. In the same way $R_p(h_3, v_i) = \delta_i A^2(A^2-9)(D-3)[(D-3)(2D-1)(3D-4)]^{i-3}Ms_i,\ i = \overline{5,7}$, where $\delta_i \neq 0, s_i,\ i = \overline{5,7}$, are polynomials in A, D, M with integer coefficients. Here is $s_i = 0, i = \overline{5,7}$, too. Let us notice that r_4, r_5 are polynomials of 5^{th} degree relative to M, s_5, r_7 of 7^{th} degree, r_6, s_6, s_7 are of $6^{th}, 9^{th}, 11^{th}$ degree, respectively. While computing the resultant of polynomials $r_4, Sr_5 + r_6$ relative to M we get

$$\begin{aligned} R_M(r_4, Sr_5 + r_6) = {} & \alpha A^{25}(A^2-6)^3(2A^2-21)^4(D-3)^{13}(D-2)(D+2)^4(2D-1)^{19}\times\\ & (3D-4)^4(4D-17)(4D-7)\, H_0^2\, [4405854208A^5(D-3)^6(2D-1)^5(3D-4)^4T_0+\\ & 196689920A^4(D-3)^5(2D-1)^4(3D-4)^3]T_1S - 26342400A^3(D-3)^3(2D-1)^3\times\\ & (3D-4)^2T_2s^2 + 3528000A^2(D-3)^2(2D-1)^2(3D-4)T_3s^3 - 472500A(D-3)\times\\ & (2D-1)T_4s^4 - 253125H_1T_5s^5], \end{aligned}$$

where $\alpha \neq 0$, $H_1 = (A^2-6)[28A^4(2D-1)(3D-4) - 6A^2(2596D^2 - 9316D + 9459) + 9(13298D^2 - 62623D + 80687)] - 81(D-3)(622D-1481)$, $H_0, T_i,\ i = \overline{0,5}$, are polynomials in A, D with integer coefficients. If $2A^2 - 21 = 0$, then from $r_4 = 0, s_5 = 0, s_6 = 0$ we have

$$A^2(D-3)^{14}(D-2)(D+2)^4(2D-1)^{18}(3D+1)^2(4D-17)(4D-7) = 0.$$

Hence, $(3D+1)(4D-17)(4D-7 = 0)$. Since

$$I_7+\langle 2A^2-21, B+3D\rangle = \langle 2A^2-21, B+3D, B^{11}P, P(104763912686092943360AB^9- 13155467622955889887953MP^3), 186824475M(413P^3+13056M^2), ..., AB+3K\rangle,$$
$$I_7+\langle 2A^2-21, 17B-4D\rangle = \langle 2A^2-21, 17B-4D, B^7(33B^2-8P)(9B^2-2P), -71876935680M^3-7AB(33B^2-8P)(9B^2-2P)(102952372707B^4- 47325572912B^2P+5362662080P^2), \dots, 3AB-2K\rangle,$$
$$I_7+\langle 2A^2-21, 7B-4D\rangle = \langle 2A^2-21, 7B-4D, B^7(5B^2-8P)(B^2+2P), AB(5B^2-8P)(B^2+2P)(56056383B^4-124509840B^2P+66718400P^2)- 98014003200M^3, \dots, AB-2K\rangle,$$

then when $2A^2-21=0$ and (27) holds, $O(0,0)$ of system (6) is a focus. If $AB\neq 0$, $(17B-4D)(7B-4D)=0$, then the study of the system of equations $g_i=0$, $i=\overline{1,7}$, shows that if (27) is fulfilled, $O(0,0)$ of system (6) can be a centre only in the case $2A^2-21=0$. So, when $(17B-4D)(7B-4D)=0$ we have the case of focus.

Let us examine now the case $H_0=0$. To do this we find

$$R_M(h_3,v_i)=\mu_iA^{30}(A^2-6)^3(2D-1)^{17}(D-3)^{10}(D-2)(D+2)^4(4D-17)\times (4D-7)[(2A^2-21)^2(D-3)^4(2D-1)^3]^{i-4}B_i,\ i=\overline{5,6},$$

where $\mu_i\neq 0$, $B_5=T_5C_1$, $B_6=T_0C_2$, C_1 and C_2 are polynomials in A,D, which consist of 1273 and 2088 factors, respectively. Then we find

$$R_A(H_0,B_i)=\lambda_i(D-3)^{14}(2D-1)^{59}(3D-4)^6(3D+1)^2(4D-7)^2(4D^2+36D- 69)^2(14D^2-19D-24)^4(16D^2-21D-6)^2(632D^3-3408D^2+6159D-3994)\times (1456D^3-11244D^2+28752D-25247)(88D^4-410D^3+993D^2-1792D+1256)\times (2184D^5-7700D^4-19135D^3+102085D^2-98789D-3402)E_i,\quad i=5,6,$$

where $\lambda_i\neq 0$, E_i, $i=5,6$, are coprime polynomials in D. Since $(D-3)(2D-1)(3D-4)(3D+1)(4D-7)\neq 0$ only in the case $(2A^2-21)(A^2-6)=0$, $R_A(H_0,B_i)=0$, $i=5,6$, we can conclude that also when $H_0=0$, $O(0,0)$ of system (6) cannot be a centre. In the case when $H_1=0$ the study of system $H_1=0$, $T_i=0$, $i=\overline{0,4}$, shows that $O(0,0)$ of system (6) is a focus. So, in the case when $B=1$ and (27) holds, $O(0,0)$ of system (6) can be a centre only when

$$T_i=0,\ i=\overline{0,5}. \tag{29}$$

Let us find the real solutions of system (29), where $A^2<8$. We have

$$R_z(T_s,T_i)=\gamma_{i,0}(D-3)^{24}(D+2)^3(2D-1)^{22}(3D-4)^2\Theta_0B_{i,0},\ i=\overline{0,4},$$

where $z=A^2$, $\gamma_{i,0}\neq 0$, Θ_0 are polynomials in D of 66^{th} degree whose coefficients are coprime integer numbers of the order from 10^{77} to 10^{114}, $B_{i,0}$, $i=\overline{0,4}$, are coprime polynomials in D of degree 690, 668, 657, 635, 613, respectively. Notice that the polynomial Θ_0 has 20 real roots.

On the other hand,

$$R_D(T_5,T_i)=\gamma_{i,1}(A^2-9)^3(A^2-6)^2(7A^2-30)^{12}\Theta^3B_{i,1},\ i\overline{=0,4},$$

where $\gamma_{i,1}\neq 0$, $B_{i,1}$, $i=\overline{0,4}$, are coprime polynomials in A, Θ is a polynomial in A of 44^{th} degree which consists of terms in even degrees and whose coefficients are coprime integer numbers of the order from 10^{32} to 10^{54}.

Let us introduce the vector $q = (A, D)$. The system (29) has 18 real solutions $q = q_i$, where $q_i = (A_i, D_i),\ i = \overline{1,9},\ q_{i+9} = (-A_i, D_i),\ i = \overline{1,9}$, and

$$A_1 = 2.48741...,\ A_2 = A_3 = A_4 = 2.495479\ldots,\ A_5 = A_6 = A_7 = 2.189944\ldots,$$
$$A_8 = 2.072126\ldots,\ A_9 = 1.916074\ldots,\ D_1 = 2.027444\ldots,\ D_2 = 2.9540003\ldots,$$
$$D_3 = 2.990356\ldots,\ D_4 = 3.008036\ldots,\ D_5 = 0.617227\ldots,\ D_6 = 1.954363\ldots,$$
$$D_7 = 5.659333\ldots,\ D_8 = 4.479633\ldots,\ D_9 = 3.057486\ldots.$$

Replacing $q_i,\ i = \overline{1,18}$, by $r_i,\ i = \overline{4,6}$, which was found from the system of equations $r_i = 0,\ i = \overline{4,6}$, find $M = M_i,\ i = \overline{1,18}$.

Then from $v_4 = 0$ we find $p = p_i,\ i = \overline{1,18}$. Consider $r = (A, D, K, L, M, P)$. Taking into account K, L which were found before, when $B = 1$, we have 18 real solutions $r = r_k = (A_k, D_k, K_k, L_k, M_k, P_k),\ k = \overline{1,18}$, of the system of equations $g_i = 0,\ i = \overline{1,6}$. Here $r_{i+9} = (-A_i, D_i, -K_i, L_i, -M_i, P_i),\ i = \overline{1,9}$. Notice that A_i are roots of the polynomial Θ, D_i are roots of the polynomial θ_0.

Let us show that $g_7|_{r=r_k} \neq 0,\ k = \overline{1,18}$. We have

$$R_M(r_4, r_7) = \alpha_0 A^{35}(A^2-6)^3(2A^2-21)^5(D-3)^{24}(D-2)(D+2)^4(2D-1)^{29}\times$$
$$(3D-4)^{12}(4D-17)(4D-7)H_0^2T_6,$$

where $\alpha_0 \neq 0$. Then we find $R_{A^2}(T_5, T_6) = \gamma_{5,6}(D-3)^{30}(D+2)(2D-1)^{26}(3D-4)^2C_0$, where $\gamma_{5,6} \neq 0$, C_0, is a polynomial in D of 997^{th} degree whose coefficients are coprime integer numbers of the order from 10^{3009} to 10^{3580}. Since Θ_0, C_0 are coprime polynomials in D, then $v_7|_{r=r_k} \neq 0,\ k = \overline{1,18}$. So, when (29) is fulfilled, $O(0,0)$ cannot be a centre.

Proof of Theorem 5. The proof follows directly from Lemmas 1–4.

Proposition 2. *When $r = r_k,\ k = \overline{1,18}$, $B = 1$, the critical point $O(0,0)$ of system (6) is a focus of 7^{th} order.*

Proof. The proof follows from Lemma 4.

Theorem 6. *For any $\varepsilon > 0,\ \delta > 0,\ k,\ (k = \overline{1,18})$ there exists $r \in U_\delta(r_k)$, where $U_\delta(r_k)$ is a δ-neighbourhood of r_k, such that system (6) with $B = 1$ has in ε-neighbourhood $U_\varepsilon(0)$ of the point $O(0,0)$ 6 limit cycles.*

Proof. The proof is analogous to the proof of Theorem 3 from [25], using Lemma 1 from [25].

References

[1] Черкас Л.А., *Об условиях центра для некоторых уравнений вида $yy' = P(x) + Q(x)y + R(x)y^2$* . Дифференц. уравнения, 1972, **8**, №8, 1435-1439.

[2] Черкас Л.А., *Условия центра для одного уравнения Льенара* . Дифференц. уравнения, 1976, **12**, №2, 292-298.

[3] Черкас Л.А., *Условия центра для уравнения $yy' = \sum_{i=0}^{3} p_i(x)y^i$*. Дифференц. уравнения, 1978, **14**, №9, 1594 - 1600.

[4] Амелькин В.В., Лукашевич Н.А., Садовский А.П., *Нелинейные колебания в системах второго порядка.* Минск: Изд-во БГУ, 1982.

[5] Gasull A., Torregrosa J., *Center Problem for Several Differential Equations via Cherkas' Method.* Journal of Mathematical Analysis and Applications, 1998, **228**, 322-343.

[6] Андреев А.Ф., *Особые точки дифференциальных уравнений.* Минск: Вышэйшая школа, 1979.

[7] Садовский А.П., *Элементарное доказательство и применения одной теоремы Ф.Такенса.* Дифференц. уравнения, 1980, **16**, №12, 2284 - 2287.

[8] Садовский А.П., *Решение проблемы центра и фокуса для некоторых систем нелинейных колебаний.* Дифференц. уравнения, 1978, **14**, №2, 380 - 382.

[9] Куклес И.С., *О некоторых случаях отличия фокуса и центра.* ДАН СССР, 1944, **42**, №5, 208 - 211.

[10] Сибирский К.С., *Алгебраические инварианты дифференциальных уравнений и матриц.* Кишинёв: Штиинца, 1976.

[11] Данилюк В.И., Шубэ А.С. *Различение случаев центра и фокуса для кубических систем с шестью параметрами.* Известия АН ССРМ. Математика, 1990, **3**, 18-21.

[12] Zoladek H., *The classification of reversible cubic systems with center.* Topological Methods in Nonlinear Analysis. Journal of the Juliusz Schauder Center, 1994, **4**, 79 - 136.

[13] Zoladek H., *Remarks on the " Classification of reversible cubic systems with center".* Topological Methods in Nonlinear Analysis. Journal of the Julius Schauder Center, 1996, **8**, 335 - 342.

[14] Rousseau Ch., Scholmiuk D., Thibadeau P., *The centres in the reduced Kukles system.* Nonlinearity, 1995, **8**, 541 - 569.

[15] Садовский А.П., *Решение проблемы центра и фокуса для кубической системы нелинейных колебаний.* Дифференц. уравнения, 1997, **33**, №2, 236 - 244.

[16] Садовский А.П., *Об условиях центра и предельных циклах одной кубической системы дифференциальных уравнений.* Дифференц. уравнения, 2000, **36**, №1, 98 - 102.

[17] Садовский А.П., *Центры и фокусы одного класса кубических систем.* Дифференц. уравнения, 2000 **36**, №12, 1652 - 1657.

[18] Lloyd N.G., Pearson J.M., *Computing centre condition for certain cubic systems.* J. Comp. Appl. Math, 1992, **40**, 323- 336.

[19] Lloyd N.G., Pearson J.M., *Five limit cycles for a simple cubic system* . Publications Matematiques, 1997, **41**, 199 - 208.

[20] Romanovski V.G., Suba A., *Centers of some cubic systems.* Ann.of Diff. Eqs, 2001, **17**:4, 363 - 376

[21] Романовский В.Г., Щеглова Н.Л., *Условия интегрируемости двух кубических векторных полей.* Дифференц. уравнения, 2000, **36**, №1, 94 - 97

[22] Cozma D., Suba A., *The solution of the problem of center for cubic differential systems with four invariant straight lines.* Analele Ştiinţifice ale Universităţii "Al.I.Cuza" Iaşi, 1998, **XLIV**.

[23] Cozma D., *Solution of the problem of the centre for a cubic differential system with three invariant straight lines.* Qualitative theory of dynamical systems, 2001, **2**, 129 - 143.

[24] Cox D., Little J., O'Shea D., *Ideals, Varieties, and Algorithms.* Springer-Verlag, New-Berlin-Hamburg, 1998.

[25] Ле Ван Линь, Садовский А.П., *Проблема центра и фокуса для одной кубической системы в случае нулевых собственных значений линейной части.* Вестник БГУ. Математика и информатика, 2002, **3**, 75 - 80.

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