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Studying stability of the equilibrium solutions in the restricted Newton's problem of four bodies

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Abstract. Newton's restricted problem of four bodies is investigated. It has been shown that there are six equilibrium solutions of the equations of motion. Stability of these solutions is analyzed in linear approximation with computer algebra system *Mathematica*. It has been proved that four radial solutions are unstable while two bisector solutions are stable if the mass of the central body P_0 is large enough. There is also a domain of instability of the bisector solutions near the resonant point in the space of parameters and its boundaries are found in linear approximation.

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1 Introduction

The main problem of the dynamics is to investigate all possible motions of a system. In the case of the system of point particles moving under their mutual gravitational force this problem has been solved only for two particles. Although there are ten integrals for such systems, the general solution of the differential equations of motion in the case of three or more interacting particles can not be obtained. So further progress in this field seems to be connected with seeking and studying particular solutions of the equations of motion. In the case of three particles five particular solutions were found by L.Euler (1767) and J.L.Lagrange (1772) [1]. To study the stability of these solutions it turned out to be necessary to elaborate new qualitative, analytical and numerical methods for studying nonlinear Hamiltonian systems [2]. Nevertheless, the elaboration of the stability theory of Hamiltonian systems has not been completed yet and the investigations in this field are very topical.

In [3–5] it was proved that there is a new class of the exact particular solutions of the planar Newton's many-body problem. On this basis two new dynamical models were proposed that are known as Newton's restricted problems of $(n+2)$ bodies [6, 7]. Now it is necessary to find all equilibrium solutions in these problems and to investigate their stability. As in general case this problem is very complicated let us start with the case of four bodies. But even in this case the calculations are very complicated and can not be done without computer. So we have used here computer algebra system *Mathematica* that is a very powerful tool for doing both analytical and numerical calculations [8].

2 Equilibrium solutions of the equations of motion

Let two point particles P_1 and P_2 of equal masses m move in elliptical orbits about their common center of mass where the third particle P_0 of mass m_0 is resting. The particles attract each other according to Newton's law of gravitation. At any instant of time the particles P_1 and P_2 are symmetrical with respect to the particle P_0 and their orbits are situated in the xOy plane of the barycentric inertial frame of reference. Using cylindrical coordinates we can write a solution of the corresponding three-body problem in the form [5]

$$\rho_j(\nu) = \frac{p}{1+e\cos\nu}, \;\; \varphi_j(t) = \nu(t) + \pi j, \;\; z_j(t) = 0 \quad (j=1,2), \tag{1}$$

where p and e are parameter and eccentricity of the elliptic orbit of the particles. The functions $\rho_j(t)$ and $\nu(t)$ are connected by the relation

$$\rho_j^2 \frac{d\nu}{dt} = \sqrt{fp(m_0 + m/4)} \equiv c, \tag{2}$$

where f is the constant of gravitation.

Let us consider the motion of the fourth particle P_3 of negligible mass m_1 in the gravitational field generated by the particles P_0, P_1 and P_2. Denoting its cylindrical coordinates as ρ, φ, z we can write Lagrangian of the system in the form

$$L = \frac{m_1}{2}(\dot\rho^2 + \rho^2\dot\varphi^2 + \dot z^2) + fm_1\left(\frac{m_0}{r} + \frac{m}{r_1} + \frac{m}{r_2}\right), \tag{3}$$

where

$$r = \sqrt{\rho^2 + z^2}, \quad r_j = \sqrt{\rho^2 + \rho_j^2 - 2\rho\rho_j\cos(\varphi - \varphi_j) + z^2} \quad (j=1,2)$$

are the distances between the particle P_3 and particles P_0, P_1, P_2, respectively, and the dot denotes the derivative $\frac{d}{dt}$. With Lagrangian (3) the equations of motion of the particle P_3 may be written as

$$\begin{gathered}
\ddot\rho - \rho\dot\varphi^2 + fm_0\frac{\rho}{r^3} + fm\frac{\rho - \rho_1\cos(\varphi-\varphi_1)}{r_1^3} + fm\frac{\rho - \rho_2\cos(\varphi-\varphi_2)}{r_2^3} = 0,\\
\rho\ddot\varphi + 2\dot\rho\dot\varphi + fm\frac{\rho_1\sin(\varphi-\varphi_1)}{r_1^3} + fm\frac{\rho_2\sin(\varphi-\varphi_2)}{r_2^3} = 0,\\
\ddot z + fm_0\frac{z}{r^3} + fm\frac{z}{r_1^3} + fm\frac{z}{r_2^3} = 0.
\end{gathered} \tag{4}$$

Taking into account (1) let us make a substitution in (4) according to the rule

$$\rho_j(t) \to \frac{p}{1+e\cos\nu}, \qquad \rho(t) \to \frac{p}{1+e\cos\nu}\,\rho(\nu),$$
$$z(t) \to \frac{p}{1+e\cos\nu}\,z(\nu), \qquad \varphi(t) \to \nu + \varphi(\nu).$$

It means that we'll consider the motion of the particle in the frame of reference rotating about Oz axis where all distances are pulsating so that the particles P_1 and P_2 are resting on the Ox axis at the points $x = \mp 1$, respectively. Such frame is known as Nechvil's configurational space [1]. Besides, we use the polar angle ν determining the position of the particles P_1 and P_2 on the xOy plane of inertial frame of reference as a new independent variable. Then derivatives of the coordinates ρ and φ are transformed as

$$\frac{d\rho}{dt} \to \frac{c}{p}\ ((1+e\cos\nu)\frac{d\rho}{d\nu} + e\sin\nu\ \rho),$$

$$\frac{d\varphi}{dt} \to \frac{c}{p^2}\ (1+e\cos\nu)^2(1+\frac{d\varphi}{d\nu}),$$

$$\frac{d^2\rho}{dt^2} \to \frac{c^2}{p^3}\ (1+e\cos\nu)^2((1+e\cos\nu)\frac{d^2\rho}{d\nu^2} + e\cos\nu\ \rho),$$

$$\frac{d^2\varphi}{dt^2} \to \frac{c^2}{p^4}\ (1+e\cos\nu)^3((1+e\cos\nu)\frac{d^2\varphi}{d\nu^2} - 2e\sin\nu\ \frac{d\varphi}{d\nu} - 2e\sin\nu).$$

Derivatives of the coordinate z are obtained from the corresponding derivatives of ρ with the substitution $\rho \to z$. Then equations of motion (4) become

$$\begin{gathered}
\frac{d^2\rho}{d\nu^2} - \rho\left(\frac{d\varphi}{d\nu}+1\right)^2 + \frac{e\cos\nu}{1+e\cos\nu}\ \rho = \\
= -\frac{4}{(1+4\mu)(1+e\cos\nu)}\left(\frac{\mu\ \rho}{(\rho^2+z^2)^{3/2}} + \frac{\rho+\cos\varphi}{r_1^3} + \frac{\rho-\cos\varphi}{r_2^3}\right), \\
\rho\frac{d^2\varphi}{d\nu^2} + 2\frac{d\rho}{d\nu}\left(\frac{d\varphi}{d\nu}+1\right) = \frac{4\sin\varphi}{(1+4\mu)(1+e\cos\nu)}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right), \\
\frac{d^2z}{d\nu^2} + \frac{e\cos\nu}{1+e\cos\nu}\ z = -\frac{4z}{(1+4\mu)(1+e\cos\nu)}\left(\frac{\mu}{(\rho^2+z^2)^{3/2}} + \frac{1}{r_1^3} + \frac{1}{r_2^3}\right),
\end{gathered} \tag{5}$$

where

$$r_1 = \sqrt{\rho^2+1+2\rho\ \cos\varphi+z^2},\ r_2 = \sqrt{\rho^2+1-2\rho\ \cos\varphi+z^2}$$

and $\mu = \frac{m_0}{m}$.

The equilibrium solutions of the system (5) are determined from the condition that all derivatives are equal to zero. The only such solution satisfying the third equation of (5) is $z = 0$. In this case the second equation of (5) may be written as

$$\sin\varphi\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) = 0.$$

It has four solutions

$$\varphi = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}.$$

For $\varphi = 0,\ \pi$ the equilibrium positions of the particle P_3 are on the straight line P_1P_2. In terms of [7] such solutions are called the radial equilibrium solutions. For $\varphi = \frac{\pi}{2},\ \frac{3\pi}{2}$ the equilibrium positions are on the straight line being perpendicular to the line P_1P_2 and are called the bisector equilibrium solutions. Substituting solutions $\varphi = 0,\ \pi$, $z = 0$ and $\rho = R = const$ into the first equation of (5) we obtain an equation determining the radial equilibrium positions

$$\mu\left(R - \frac{1}{R^2}\right) + \frac{R}{4} - \left(\frac{R-1}{|R-1|^3} + \frac{R+1}{|R+1|^3}\right) = 0. \tag{6}$$

The corresponding equation determining bisector equilibrium positions is

$$\mu\left(R - \frac{1}{R^2}\right) + \frac{R}{4} - \frac{2R}{(R^2+1)^{3/2}} = 0. \tag{7}$$

If $m_0 = 0$ then equations (6), (7) coincide with the corresponding equations determining positions of the points of libration in the restricted problem of three bodies [1, 2]. In this case equation (6) has two solutions. One solution is $R = 0$ and another one is such a root of the equation

$$R = \frac{8(R^2+1)}{(R^2-1)^2}$$

that satisfies the condition $R > 1$. Equation (7) has also two solutions: $R = 0$ and $R = \sqrt{3}$. The second solution determines two equilibrium positions of the particle P_3 being symmetrical with respect to the origin that correspond to the famous Lagrange's triangular solutions. It is known that all solutions above are unstable in the sense of Liapunov [2]. So further on we'll consider the case $m_0 \neq 0$. Analyzing equation (6) one can conclude that in the domain $0 \leq R < 1$ it can be rewritten as

$$\mu\left(R - \frac{1}{R^2}\right) + \frac{R}{4} + \frac{4R}{(1-R^2)^2} = 0 \tag{8}$$

and has only one root. For $R > 1$ equation (6) has the form

$$\mu\left(R - \frac{1}{R^2}\right) + \frac{R}{4} - \frac{2(R^2+1)}{(R^2-1)^2} = 0. \tag{9}$$

This equation also has one root. It should be noticed that the roots of equations (8), (9) tend to a limit $R = 1$ as $\mu \to \infty$. Equation (7) has only one root and its value decreases from $R = \sqrt{3}$ to $R = 1$ as parameter μ tends to infinity. Thus, in the case $m_0 \neq 0$ there are six equilibrium solutions of the restricted problem of four bodies in Nechvil's configurational space. Four of them are the radial equilibrium solutions $\varphi = 0,\ \pi$, $z = 0$ and the corresponding values of R are given as roots of equations (8), (9). The last two solutions form a couple of the bisector equilibrium solutions $\varphi = \frac{\pi}{2},\ \frac{3\pi}{2}$, $z = 0$ and R is given as a root of equation (7).

3 Studying stability of the equilibrium solutions

The stability problem of the equilibrium solutions found above is connected with the investigation of nonlinear differential equations of the disturbed motion. Usually, the first step in solving this problem is an analysis of the corresponding linearized system. In order to investigate equations of motion (5) in the vicinity of the equilibrium solutions let us make in (5) the substitution

$$\rho(\nu) \to R + u(\nu), \ \ \varphi(\nu) \to \beta + \gamma(\nu).$$

Considering the functions $u(\nu)$, $\gamma(\nu)$, $z(\nu)$ as small perturbations of the equilibrium solutions we can expand equations (5) in Taylor series in powers of u, γ and z and neglect all terms of the second and higher orders. Then we obtain equations linearized in the vicinity of the radial equilibrium solutions in the form

$$\begin{aligned}
&\frac{d^2u}{d\nu^2} - 2R\frac{d\gamma}{d\nu} = \frac{3+2a_j}{1+e\cos\nu}u, \\
&\frac{d^2\gamma}{d\nu^2} + \frac{2}{R}\frac{du}{d\nu} = -\frac{a_j}{1+e\cos\nu}\gamma, \\
&\frac{d^2z}{d\nu^2} + \frac{1+a_j+e\cos\nu}{1+e\cos\nu}z = 0,
\end{aligned} \tag{10}$$

where

$$a_1 = \frac{8(R^2+3)}{(1+4\mu)(1-R^2)^3}, \quad a_2 = \frac{8(3R^2+1)}{(1+4\mu)(R^2-1)^3}.$$

The corresponding system of equations linearized in the vicinity of the bisector equilibrium solutions is

$$\begin{aligned}
&\frac{d^2u}{d\nu^2} - 2R\frac{d\gamma}{d\nu} = \frac{3-b}{1+e\cos\nu}u, \\
&\frac{d^2\gamma}{d\nu^2} + \frac{2}{R}\frac{du}{d\nu} = \frac{b}{1+e\cos\nu}\gamma, \\
&\frac{d^2z}{d\nu^2} + z = 0,
\end{aligned} \tag{11}$$

where

$$b = \frac{24}{(1+4\mu)(1+R^2)^{5/2}}.$$

Let us note that parameters R and μ are connected by the relations (7)-(9). So the constants a_1, a_2 and b depend only on the parameter μ.

Thus, we have obtained two systems of three linear differential equations of the second order with periodic coefficients. It is evident that coefficients of equations (10),(11) are analytic functions of parameter e in the domain $|e| < 1$. Consequently, the behavior of the solutions of these systems is determined by their characteristic exponents calculated for $e = 0$. If the system has at least one characteristic exponent with positive real part for $e = 0$, then it is unstable for sufficiently small $e \neq 0$. If all characteristic exponents of the system are complex numbers with unit magnitude but some of them are multiple, then the system is unstable, too. But if all characteristic exponents of the system are different and pure imaginary numbers, then the instabilities can arise only when the characteristic exponents λ_k satisfy the resonance conditions

$$\lambda_k \pm \lambda_l = iN \quad (k, l = 1,\ 2,\ 3,\ 4;\ N = 0,\ \pm 1,\ \pm 2,\ ...\ ). \tag{12}$$

So we should calculate the characteristic exponents of systems (10), (11) for $e = 0$.

The third equation in systems (10), (11) is independent of the first two. It means that in the linear approximation the disturbed motion of the particle P_3 in xOy plane does not depend on its motion along the $0z$ axis. So we may analyze these motions separately. The third equation of the system (10) is just a Hill's equation. For $e = 0$ it has two pure imaginary characteristic exponents $\pm i\sqrt{1 + a_j}$ because $a_j > 0$ for any μ. It was investigated in detail in [9] where it was shown that there are the domains of instability of this equation in the vicinity of the points $a_j = \frac{(2k-1)^2}{4} - 1$ $(k = 1,\ 2,\ ...)$. Using those results and relationships (8), (9) it is easy to construct the corresponding domains of instability of the third equation of (10) in the μOe plane. Characteristic exponents of the first two equations of system (10) for $e = 0$ are calculated very easy and may be written in the form

$$\lambda_k = \pm\frac{1}{\sqrt{2}}\left(-1 + a_j \pm \sqrt{1 + 10a_j + 9a_j^2}\right)^{1/2} \quad (k = 1,\ 2,\ 3,\ 4). \tag{13}$$

Numerical calculations show that one of the characteristic exponents (13) is a positive real number for any μ from the interval $0 \le \mu < \infty$ and for both coefficients a_1 and a_2. So, according to Liapunov's theorem on linearized stability [2], we can conclude that radial equilibrium solutions of the restricted problem of four bodies are unstable.

The third equation of system (11) has two pure imaginary characteristic exponents $\pm i$ and is stable for any e. Characteristic exponents of the first two equations of system (11) for $e = 0$ can be written as

$$\lambda_k = \pm\frac{1}{\sqrt{2}}\left(-1 \pm \sqrt{1 - 12b + 4b^2}\right)^{1/2} \quad (k = 1,\ 2,\ 3,\ 4). \tag{14}$$

If the condition

$$0 < 1 - 12b + 4b^2 < 1 \tag{15}$$

is fulfilled, then characteristic exponents (14) are different and pure imaginary numbers $\lambda_k = \pm i\sigma_{1,2}$ where

$$\sigma_{1,2} = \frac{1}{\sqrt{2}}\left(1 \pm \sqrt{1 - 12b + 4b^2}\right)^{1/2}.$$

Numerical analysis shows that $1 - 12b + 4b^2 < 1$ for any μ. For $\mu = 11.7203$ the expression $1 - 12b + 4b^2$ becomes zero and the system (11) has two multiple characteristic exponents $\pm\frac{i}{\sqrt{2}}$. In this case the system is unstable for $e = 0$ because its solution has linearly growing terms of the form $\nu\cos\frac{\nu}{\sqrt{2}}$, $\nu\sin\frac{\nu}{\sqrt{2}}$. The inequality (15) is true if $11.7203 < \mu < \infty$. In this case

$$\frac{1}{\sqrt{2}} < \sigma_1 < 1, \quad 0 < \sigma_2 < \frac{1}{\sqrt{2}}.$$

So the resonance condition (12) can be fulfilled only for $\lambda_k = i/2$, $\lambda_l = -i/2$, $N = 1$ when $\sigma_2 = 1/2$ and $\mu = \mu_R = 15.9691$. For $e > 0$ in the vicinity of the resonant value μ_R of parameter μ a domain of instability can exist. To find the boundaries of this domain it is necessary to calculate the fundamental matrix of the system (11). And to do this we'll use Liapunov-Poincare method of a small parameter.

The first two equations of system (11) can be written in the form

$$\frac{dx}{d\nu} = P(\nu, e)x, \tag{16}$$

where x is a vector with four components and $P(\nu, e)$ is an 4×4 matrix function that can be represented in the form

$$P(\nu, e) = P_0 + \sum_{k=1}^{\infty} P_k(\nu)e^k, \tag{17}$$

and

$$P_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -2/R & 0 & 0 & 1/R^2 \\ -(1+b) & 0 & 0 & 2/R^2 \\ 0 & bR^2 & 0 & 0 \end{pmatrix}, \quad P_k(\nu) = (-\cos\nu)^k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3-b & 0 & 0 & 0 \\ 0 & bR^2 & 0 & 0 \end{pmatrix}.$$

The series (17) converges for any ν in the domain $|e| < 1$ and $P_k(\nu)$ are continuous finite functions. According to Liapunov theorem [10, 11], the fundamental matrix $X(\nu, e)$ for the system (16) normalized by the condition $X(0, e) = E_4$, where E_4 is an 4×4 identity matrix, may be represented in the form

$$X(\nu, e) = \exp(P_0\nu)Z(\nu, e)\exp(\nu W(e)), \tag{18}$$

where $Z(\nu, e) = Z(\nu + 2\pi, e)$ is a periodic analytic matrix function and $W(e)$ is a constant matrix. The matrices $Z(\nu, e)$ and $W(e)$ may be also represented in the form of series in powers of e

$$Z(\nu, e) = \sum_{k=0}^{\infty} Z_k(\nu)e^k, \; Z_0(0) = E_4, \; Z_k(0) = 0 \; (k \geq 1), \tag{19}$$

$$W(e) = \sum_{k=1}^{\infty} W_k\, e^k. \tag{20}$$

The series (19), (20) converge in the domain $|e| < 1$ for any ν and $Z_k(\nu)$ are continuous matrices satisfying the next recurrence relation

$$\frac{dZ_k}{d\nu} = \sum_{l=1}^{k} (\exp(-P_0\nu)P_l(\nu)\exp(P_0\nu)Z_{k-l} - Z_{k-l}W_l)\,. \tag{21}$$

Matrices W_k can be found from the condition that $Z_k(\nu)$ are periodic matrices. Actually, in the first order equation (21) has the form

$$\frac{dZ_1}{d\nu} = \exp(-P_0\nu)P_1(\nu)\exp(P_0\nu) - W_1. \tag{22}$$

Taking into account initial conditions (19) we can write a solution of equation (22) as

$$Z_1(\nu) = -W_1\nu + \int_0^{\nu} \exp(-P_0\tau)P_1(\tau)\exp(P_0\tau)d\tau.$$

Using periodicity of the matrix $Z_1(\nu)$ we obtain

$$W_1 = \frac{1}{2\pi}\int_0^{2\pi} \exp(-P_0\tau)P_1(\tau)\exp(P_0\tau)d\tau.$$

Calculations in the higher orders are done in a similar way. But with the k growth they become more and more cumbersome and can not be done without computer. Here we have calculated the fundamental matrix $X(\nu, e)$ in the vicinity of the resonant point $\mu_R = 15.9691$ with the accuracy $o(e^2)$. Then we can write the characteristic equation for the system (16) as

$$\begin{gathered}
\rho^4 + \rho^3(2 - 2\cos(\sqrt{3}\pi) - \frac{8}{\sqrt{3}}\,e\pi s_1\sin(\sqrt{3}\pi)+ \\
+\rho^2(2 - 4\cos(\sqrt{3}\pi) - \frac{16}{\sqrt{3}}\,e\pi s_1\sin(\sqrt{3}\pi) + \frac{e^2}{48}(\pi^2(-99 + 1024s_1^2)- \\
-6\cos^2(\sqrt{3}\pi/2)(297 + 215\cos(\sqrt{3}\pi))))+ \\
+\rho(2 - 2\cos(\sqrt{3}\pi) - \frac{8}{\sqrt{3}}e\pi s_1\sin(\sqrt{3}\pi) + \frac{e^2}{24}(\pi^2(99\cos(\sqrt{3}\pi)- \\
-256s_1^2(-1 + 3cos(\sqrt{3}\pi))) - 123\sin^2(\sqrt{3}\pi)))+ \\
+1 + \frac{e^2}{96}(2181 + 2\pi^2(-99 + 1024s_1^2) + 3072\cos(\sqrt{3}\pi) + 891\cos(2\sqrt{3}\pi)) = 0,
\end{gathered} \tag{23}$$

where s_1 is a small parameter determining deviation of σ_2 from its resonant value according to the relation $\sigma_2 = \frac{1}{2} + s_1 e$. Analysis of equation (23) shows that in the vicinity of the resonant value of parameter $\sigma_2 = \frac{1}{2}$ there is a domain in the $\sigma_2 e$ plane where the stability condition $|\rho| \leq 1$ is not fulfilled. This domain is bounded by the straight lines $\sigma_2 = \frac{1}{2} \pm \frac{\sqrt{33}}{16}\,e$. The corresponding domain of instability in the μe plane is bounded by the straight lines

$$\mu = 15.9691 \pm 16.2652\ e. \tag{24}$$

Thus, we can formulate the next theorems.

Theorem 1. *The radial equilibrium solutions of the restricted problem of four bodies are unstable for sufficiently small* e *and any values of parameter* μ.

Theorem 2. *The bisector equilibrium solutions of the restricted problem of four bodies are stable in linear approximation for* $e = 0$ *if parameter* μ *satisfies the next inequality:* $11.7203 < \mu < \infty$. *For sufficiently small values of* e *there is a domain of instability in the* μe *plane between the straight lines defined in (24).*

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