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Algebraic equations with invariant coefficients in qualitative study of the polynomial homogeneous differential systems*

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Abstract. For planar polynomial homogeneous real vector field $X = (P, Q)$ with $\deg(P) = \deg(Q) = n$ some algebraic equations of degree $n+1$ with $GL(2, \mathbb{R})$-invariant coefficients are constructed. A recurrent method for the construction of these coefficients is given. In the generic case each real or imaginary solution s_i $(i = 1, 2, \ldots, n+1)$ of the main equation is a value of the derivative of the slope function, calculated for the corresponding invariant line. Other constructed equations have, respectively, the solutions $1/s_i$, $1 - s_i$, $s_i/(s_i - 1)$, $(s_i - 1)/s_i$, $1/(1 - s_i)$. The equation with the solutions $(n+1)s_i - 1$ is called residual equation. If X has real invariant lines, the values and signs of solutions of constructed equations determine the behavior of the orbits in a neighbourhood at infinity. If X has not real invariant lines, it is shown that the necessary and sufficient conditions for the center existence can be expressed through the coefficients of residual equation.

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1 The homogeneous differential system

Let $n \geq 1$ be a positive integer, $x, y : \mathbb{R} \to \mathbb{R}$ be some unknown functions of real variable t such that $x = x(t)$, $y = y(t)$, $(\forall)\ t \in \mathbb{R}$, $a_{i,j}$, $b_{i,j}$ be real numbers for all positive integers i and j with $i + j = n$, $C_n^k = \binom{n}{k}$ be the binomial coefficients for every positive integer k, $0 \leq k \leq n$.

Let us consider the polynomial homogeneous differential system

$$\frac{dx}{dt} = \sum_{k=0}^{n} C_n^k a_{n-k,k} x^{n-k} y^k = P_n(x, y),$$
$$\frac{dy}{dt} = \sum_{k=0}^{n} C_n^k b_{n-k,k} x^{n-k} y^k = Q_n(x, y). \tag{1}$$

Let $GL(2, \mathbb{R})$ be the group of non-degenerate linear homogeneous transformations. It is known that the homogeneous polynomials $P_n(x, y)$ and $Q_n(x, y)$ are relatively prime iff the resultant μ_n of these polynomials is not equal to zero.

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Remark 1. *The resultant μ_n is a GL-invariant of the degree $2n$ with respect to the system (1) coefficients and with the weight equal to n^2-n.*

Remark 2. *The homogeneous polynomial $F_{n+1}(x,y)=yP_n(x,y)-xQ_n(x,y)$ is a GL-comitant of the degree $n+1$ with respect to variables x and y, of the degree 1 with respect to the system (1) coefficients and with the weight equal to -1. The nontrivial solutions of the equation $F_{n+1}(x,y)=0$ determine the system (1) invariant straight lines (real or imaginary).*

We suppose that

$$\mu_n = Res\,(P_n,Q_n)\neq 0,\quad F_{n+1}(x,y)=yP_n(x,y)-xQ_n(x,y)\not\equiv 0 \tag{2}$$

and denote the following polynomials and functions:

$$G_{n+1}(x,y)=xP_n(x,y)+yQ_n(x,y),\quad T_{n-1}(x,y)=\frac{\partial P_n(x,y)}{\partial x}+\frac{\partial Q_n(x,y)}{\partial y},$$
$$\varphi:\ \mathbb{C}\setminus E_\varphi\to\mathbb{C},\ \varphi(1,k)=\frac{Q_n(1,k)}{P_n(1,k)},\quad \psi:\ \mathbb{C}\setminus E_\psi\to\mathbb{C},\ \psi(s,1)=\frac{P_n(s,1)}{Q_n(s,1)}, \tag{3}$$

where $E_\varphi=\{k\mid k\in\mathbb{C},\ P_n(1,k)=0\}$ and $E_\psi=\{s\mid s\in\mathbb{C},\ Q_n(s,1)=0\}$. The functions φ and ψ are called the slope functions for the system (1).

Remark 3. *The homogeneous polynomial $T_{n-1}(x,y)$ is a GL-comitant of the degree $n-1$ with respect to variables x and y, of the degree 1 with respect to the system (1) coefficients and with the weight equal to 0.*

Because the GL-comitant $F_{n+1}(x,y)$ is not equal to zero identically, then there exist constants $u_i\in\mathbb{C}$ and $v_i\in\mathbb{C}$ such that $F_{n+1}(x,y)$ has the factorization

$$F_{n+1}(x,y)=\prod_{i=1}^{n+1}(u_ix+v_iy),\quad u_i^2+v_i^2\neq 0,\quad (\forall)\ i=1,2,\dots,n,n+1. \tag{4}$$

For $v_i\neq 0$ $(u_i\neq 0)$ we denote by $k_i=-u_i/v_i$ $(s_i=-v_i/u_i)$ the roots of the equation $F_{n+1}(1,k)=0$ $(F_{n+1}(s,1)=0)$.

The discriminant D_{n+1} of the homogeneous equation $F_{n+1}(x,y)=0$ has the form

$$D_{n+1}=\prod_{1\le i<j\le n+1} d_{i,j}^2,\qquad d_{i,j}=u_iv_j-u_jv_i. \tag{5}$$

For $j\neq k$ $(k=1,2,\dots,n,n+1)$ we denote

$$f_k=(-1)^n\prod_{j=1}^{n+1} d_{k,j}. \tag{6}$$

From relations (5) and (6) follows

Proposition 1. *The discriminant* D_{n+1} *has the factorization*

$$\prod_{k=1}^{n+1} f_k = (-1)^{\frac{n(n+1)}{2}} D_{n+1}. \tag{7}$$

Remark 4. *Each* u_i *and* v_i *have the same degree* $1/(n+1)$ *and the weight equal, respectively, to* $-1/(n+1)$ *and* $n/(n+1)$. *Each* d_{ij} *has the degree* $2/(n+1)$ *and the weight equal to* $(n-1)/(n+1)$, *each* f_i *has the degree* $2n/(n+1)$ *and the weight equal to* $n(n-1)/(n+1)$. *The discriminant* D_{n+1} *is a* GL*-invariant of the degree* $2n$ *with respect to the system (1) coefficients and with the weight equal to* n^2-n.

Let $X_i = u_i x + v_i y$ be the factor i $(i = 1, 2, \ldots, n+1)$ in the factorization (4) and $X_i = 0$ be the equation of the corresponding invariant line.

Let $p = (p_1, p_2, \ldots, p_n, p_{n+1})$ and $q = (q_1, q_2, \ldots, q_n, q_{n+1})$ be two symbolic $(n+1)$ - tuples of letters. Let us consider the symbolic differential operator

$$\Omega^1_{pq} \;=\; p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2} + \ldots + p_n \frac{\partial}{\partial q_n} + p_{n+1} \frac{\partial}{\partial q_{n+1}}, \tag{8}$$

its powers $\Omega^m = \Omega^{m-1}(\Omega^1)$ for every positive integer $m \geq 2$ and $(n+1)$- tuples

$$\begin{aligned} u &= (u_1, u_2, \ldots, u_n, u_{n+1}), \qquad & v &= (v_1, v_2, \ldots, v_n, v_{n+1}), \\ f &= (f_1, f_2, \ldots, f_n, f_{n+1}), \qquad & g &= (g_1, g_2, \ldots, g_n, g_{n+1}). \end{aligned} \tag{9}$$

By using the differential operator (8) for $(n+1)$- tuples u and v from (9) by conditions (2) and (4) we obtain the following expressions for the system (1) coefficients:

$$\begin{gathered} b_{n,0} = -u_1 u_2 \ldots u_n u_{n+1}, \quad a_{0,n} = v_1 v_2 \ldots v_n v_{n+1}, \\ C_n^k \, a_{n-k,k} = C_n^{k+1} \, b_{n-k-1,k+1} + \frac{1}{(k+1)!} \Omega^{k+1}_{vu}(-b_{n,0}), \quad 0 \leq k \leq n-1. \end{gathered} \tag{10}$$

Takes place

Lemma 1. *For every* $i = 1, 2, \ldots, n, n+1$ *the relations*

$$\begin{gathered} F_{n+1}(v_i, -u_i) = 0, \quad Q_n(v_i, -u_i) = -u_i g_i, \quad P_n(v_i, -u_i) = v_i g_i, \\ \frac{\partial F_{n+1}}{\partial y}(v_i, -u_i) = v_i f_i, \quad \frac{\partial F_{n+1}}{\partial x}(v_i, -u_i) = u_i f_i, \\ G_{n+1}(v_i, -u_i) = (u_i^2 + v_i^2) g_i, \quad T_{n-1}(v_i, -u_i) = (n+1) g_i - f_i, \\ \mu_n = g_1 g_2 \cdot \ldots \cdot g_n g_{n+1}, \quad 1 - \varphi'(1, k_i) = 1 - \psi'(s_i, 1) = \frac{f_i}{g_i}, \\ \frac{1}{n!} \Omega^n_{fg}(g_1 g_2 \cdot \ldots \cdot g_n g_{n+1}) = f_1 f_2 \cdot \ldots \cdot f_n f_{n+1} \end{gathered} \tag{11}$$

hold, where

$$g_i = \sum_{k=1}^{n} (-1)^{k+1} C_n^k \, b_{n-k,k} v_i^{n-k} u_i^{k-1} + \frac{\partial(-b_{n,0})}{\partial u_i} \, v_i^n. \tag{12}$$

Proof. The first two equalities from (11) are evident. From the identity

$$F_{n+1}(v_i,-u_i)=-u_iP_n(v_i,-u_i)-v_iQ_n(v_i,-u_i)=0$$

we obtain $P_n(v_i,-u_i)=v_ig_i$. From the relations

$$\frac{\partial F_{n+1}(x,y)}{\partial x}=\sum_{i=1}^{n+1}u_i\frac{\partial F_{n+1}}{\partial X_i},\quad \frac{\partial F_{n+1}(x,y)}{\partial y}=\sum_{i+1}^{n+1}v_i\frac{\partial F_{n+1}}{\partial X_i}$$

and (6) we obtain the identities

$$\frac{\partial F_{n+1}}{\partial y}(v_i,-u_i)=v_if_i,\quad \frac{\partial F_{n+1}}{\partial x}(v_i,-u_i)=u_if_i.$$

The relation for polynomial $G_{n+1}(x,y)$ results from the second and third equalities from (11). For polynomial $T_{n-1}(x,y)$ the following representation

$$(x^2+y^2)T_{n-1}(x,y)=(n+1)G_{n+1}-x\frac{\partial F_{n+1}}{\partial y}+y\frac{\partial F_{n+1}}{\partial x}$$

holds. From the last identity for $x=v_i,\ y=-u_i$ we obtain the required relation $T_{n-1}(v_i,-u_i)=(n+1)g_i-f_i$.

From Remark 1, the obtained relations (11) and $u_i^2+v_i^2\neq 0$ it follows that each equality $g_i=0$ implies the relation $\mu_n=0$. From Remark 4, conditions (10) and (12) it results that each addendum from g_i has the weight and the degree equal, respectively, to $n(n-1)/(n+1)$ and $2n/(n+1)$. So, the product $g_1g_2\cdot\ldots\cdot g_ng_{n+1}$ has also the degree $2n$ with respect to the coefficients of the polynomials P_n and Q_n and the weight equal to n^2-n. Thus, $\mu_n=g_1g_2\cdot\ldots\cdot g_ng_{n+1}$.

Let $D_{n+1}\neq 0$. Because $\deg(F_{n+1})=\deg(T_{n-1})+2$, then for $v_i\neq 0$ or $u_i\neq 0$ we obtain, respectively, the equalities:

$$\sum_{i=1}^{n+1}\frac{T_{n-1}(1,k_i)}{(F_{n+1})'_k(1,k_i)}=0,\qquad \sum_{i=1}^{n+1}\frac{T_{n-1}(s_i,1)}{(F_{n+1})'_s(s_i,1)}=0.$$

We have

$$\frac{T_{n-1}(1,k_i)}{(F_{n+1})'_k(1,k_i)}=\frac{T_{n-1}(1,k_i)}{v_i(F_{n+1})'_{X_i}(1,k_i)}=\frac{T_{n-1}(1,-u_i/v_i)}{v_i(F_{n+1})'_{X_i}(1,-u_i/v_i)}=$$
$$\frac{T_{n-1}(v_i,-u_i)}{f_i}=\frac{(n+1)g_i-f_i}{f_i}. \tag{13}$$

Finally we obtain

$$\sum_{i=1}^{n+1}\frac{(n+1)g_i-f_i}{f_i}=0\quad\Leftrightarrow\quad\sum_{i=1}^{n+1}\frac{g_i}{f_i}=1.$$

The last equality gives us the last relation from (11). If $D_{n+1}=0$, then for some i and j $(i\neq j)$ we have $f_i=f_j=0$ and the required equality is trivial.

From the obtained relations it follows that if $\mu_n \neq 0$, then

$$P_n^2(v_i, -u_i) + Q_n^2(v_i, -u_i) = (u_i^2 + v_i^2) g_i^2 \neq 0.$$

For derivatives of the defined slope functions we obtain:

If $v_i \neq 0$, then $F_{n+1}(1, k_i) = 0$ and $P_n(1, k_i) \neq 0$. We calculate the derivative of the function $k - \varphi(1, k)$ and determine the value of this derivative for $k = k_i$:

$$1 - \varphi'(1, k) = \left[k - \frac{Q_n(1, k)}{P_n(1, k)}\right]' = \left[\frac{kP_n(1, k) - Q_n(1, k)}{P_n(1, k)}\right]' =$$

$$\left[\frac{F_{n+1}(1, k)}{P_n(1, k)}\right]' = \frac{F'_{n+1}(1, k)P_n(1, k) - F_{n+1}(1, k)P'_n(1, k)}{P_n^2(1, k)},$$

$$1 - \varphi'(1, k_i) = \frac{F'_{n+1}(1, k_i)}{P_n(1, k_i)} = \left(\frac{\partial F_{n+1}}{\partial y}(v_i, -u_i)/P_n(v_i, -u_i)\right) = \frac{v_i f_i}{v_i g_i} = \frac{f_i}{g_i}.$$

If $u_i \neq 0$, then $F_{n+1}(s_i, 1) = 0$ and $Q_n(s_i, 1) \neq 0$. We calculate the derivative of the function $s - \psi(s, 1)$ and determine the value of this derivative for $s = s_i$:

$$1 - \psi'(s, 1) = \left[s - \frac{P_n(s, 1)}{Q_n(s, 1)}\right]' = \left[\frac{sQ_n(s, 1) - P_n(s, 1)}{Q_n(s, 1)}\right]' =$$

$$-\left[\frac{F_{n+1}(s, 1)}{Q_n(s, 1)}\right]' = -\frac{F'_{n+1}(s, 1)Q_n(s, 1) - F_{n+1}(s, 1)Q'_n(s, 1)}{Q_n^2(s, 1)},$$

$$1 - \psi'(s_i, 1) = -\frac{F'_{n+1}(s_i, 1)}{Q_n(s_i, 1)} = -\left(\frac{\partial F_{n+1}}{\partial x}(v_i, -u_i)/Q_n(v_i, -u_i)\right) = \frac{u_i f_i}{u_i g_i} = \frac{f_i}{g_i}.$$

So, it follows that the values of derivatives of the functions $k - \varphi(1, k)$ and $s - \psi(s, 1)$ for the invariant line $X_i = 0$ are the same. Lemma 1 is proved.

Remark 5. *Each f_i and g_i have the same weight and the degree equal, respectively, to $n(n-1)/(n+1)$ and $2n/(n+1)$.*

From Lemma 1 and (10) we obtain the equality

$$\frac{\partial F_{n+1}}{\partial x}(v_i, -u_i) = -(n+1)b_{n,0}v_i^n + \sum_{k=1}^{n} \frac{(-1)^k}{k!}\,(n+1-k)\Omega_{vu}^k(-b_{n,0})v_i^{n-k}u^k. \quad (14)$$

2 Construction of algebraic equations with invariant coefficients

Methods of studying the behavior of the integral curves of the system (1) have been developed by many authors (see [1, 2, 5–9, 11, 14, 21, 23–26, 30, 31, 36]). Using Forster's method (in polar coordinates), Shilov's geometrical method or local charts method (traditional method) the systems (1) with $n = 1, 2, 3$ were investigated (see [10,12,19,22,26,35,40,41,44,45]). A classification of the system (1) with $n = 2$ by means of non-associative algebras was given in [16]. The algebraic and topological classifications of the system (1) with $n = 2$ by means of quadratic transformations and invariants were established in [32] and [37]. The Poincaré index method for topological classification of system (1) was applied (see [17, 18]).

The GL-comitants of the system (1) with $n = 1, 2, 3$ and the polynomial basis of these comitants have been used for algebraic, topological and geometrical classifications (see [20, 28, 29, 33, 34, 38, 39, 43, 46]).

The problem under consideration is an important step in the qualitative investigation of behavior of integral curves: at infinity for planar polynomial differential systems with maximal degree equal to n; near critical point $(0, 0)$ for planar polynomial differential systems with minimal degree equal to n. Because of this, much of the research in this area is dedicated to the investigation of the problem, usually in local charts. The simplest (but nontrivial) way of investigation is to find the algebraic classification of binary form $F_{n+1}(x, y)$ in coefficients terms (or invariant terms) and to use the results for classification of the system (1) (see [38],[44]).

Our first goal is to show that it is possible to express the conditions which delimit classes with different distributions of infinite singular points through affine invariants and comitants without knowing the basis of the affine invariants and comitants of the system (1). The second goal is to construct such invariants and comitants and to determine the geometrical significance of these objects.

In this work we develope the method of construction and show that the necessary and sufficient conditions for the center existence can be expressed through the coefficients of the residual equation. The contribution idea is due to P.Curtz paper's (see [42]) and Hilbert's symbolic operators (see [47]).

We verify our results by using Shilov's, Forster's and local charts methods for the system (1) with $n = 1, 2, 3$. The constructed invariants determine the values and the signs of the solutions and solve the problems of algebraical, topological and geometrical classifications of given systems.

For every $i = 1, 2, \dots, n + 1$ we denote

$$\xi_i = \frac{f_i}{g_i} = 1 - \varphi'(1, k_i) = 1 - \psi'(s_i, 1) \tag{15}$$

such that every ξ_i is a root of the algebraic equation

$$(g_1\xi - f_1)(g_2\xi - f_2) \cdot \ldots \cdot (g_{n+1}\xi - f_{n+1}) = 0.$$

By using the differential operator (8) for $(n + 1)$-tuples f and g from (9) the last equation can be written in the form

$$t_0\, \xi^{n+1} - t_1\, \xi^n + t_2\, \xi^{n-1} - \ldots + (-1)^n\, t_n\, \xi + (-1)^{n+1}\, t_n = 0, \tag{16}$$

where

$$t_0 = \mu_n = g_1 g_2 \cdot \ldots \cdot g_{n+1}, \quad t_i = \frac{1}{i!}\, \Omega^i_{fg}\, (\mu_n) \ \text{ for } \ (\forall)\, i = 1, 2, \dots, n,$$
$$t_n = t_{n+1} = \ (-1)^{n(n+1)/2}\, D_{n+1} = f_1 f_2 \cdot \ldots \cdot f_{n+1}. \tag{17}$$

The equation (16) will be called the main equation of the system (1).

Remark 6. *For given solution ξ_i of the main equation the equations with solutions*

$$\frac{1}{\xi_i}, \quad 1-\xi_i, \quad \frac{\xi_i}{\xi_i-1}, \quad \frac{\xi_i-1}{\xi_i}, \quad \frac{1}{1-\xi_i} \tag{18}$$

can be constructed.

For example, if we put in (16) $\xi = 1-\eta$, then obtain the equation with solutions $\eta_i = \varphi'(1,k_i) = \psi'(s_i,1)$:

$$m_0\, \eta^{n+1} - m_1\, \eta^n + m_2\, \eta^{n-1} - \ldots + (-1)^n\, m_n\, \eta + (-1)^{n+1}\, m_{n+1} = 0 \tag{19}$$

such that for every $i = 1,2,\ldots,n$ we have

$$m_0 = t_0 = \mu_n, \quad m_i = \sum_{r=0}^{i} (-1)^r\, C_{n+1-r}^{n+1-i}\, t_r, \quad m_{n+1} = \sum_{r=0}^{n-1} (-1)^r\, t_r. \tag{20}$$

Let us consider the following differential operator

$$\Theta^1 = \Omega^1_{uu} + \Omega^1_{vv}, \tag{21}$$

where u and v are from (9). It is very easy to verify that $\Theta^1(F_{n+1}) = (n+1)F_{n+1}$. So, the differential operator (21) does not change the invariant straight lines of the system (1).

From condition (2) and Euler's formulae we have two representations for the comitant $F_{n+1}(x,y)$:

$$\begin{gathered} F_{n+1}(x,y) = yP_n(x,y) - xQ_n(x,y), \\ (n+1)F_{n+1}(x,y) = y\frac{\partial F_{n+1}(x,y)}{\partial y} + x\frac{\partial F_{n+1}(x,y)}{\partial x}. \end{gathered} \tag{22}$$

It results from (22) that the differential operator (21) satisfies the relations

$$\Theta^1(P_n(x,y)) = \frac{\partial F_{n+1}(x,y)}{\partial y}, \qquad \Theta^1(Q_n(x,y)) = -\frac{\partial F_{n+1}(x,y)}{\partial x}.$$

From the last equalities we obtain the following coefficients relations:

$$\begin{gathered} \Theta^1(C_n^k a_{n-k,k}) = (k+1)(C_n^k a_{n-k,k} - C_n^{k+1} b_{n-k-1,k+1}), \\ k = 0,1,2,\ldots,n-2,n-1,\ \Theta^1(a_{0,n}) = (n+1)a_{0,n}, \\ \Theta^1(C_n^k b_{n-k,k}) = (n+1-k)(C_n^k b_{n-k,k} - C_n^{k-1} a_{n+1-k,k-1}), \\ k = 1,2,\ldots,n-1,n,\ \Theta^1(b_{n,0}) = (n+1)b_{n+1}. \end{gathered}$$

The equalities $(k+1)C_n^{k+1} = (n-k)C_n^k$ and $(n+1-k)C_n^{k-1} = kC_n^k$ imply the following rules of derivation for system's (1) coefficients:

$$\Theta^1(a_{n-k,k}) = (k+1)a_{n-k,k} - (n-k)b_{n-k-1,k+1},$$

$$k=0,1,2,\dots,n-2,n-1,\quad \Theta^1(a_{0,n})=(n+1)a_{0,n},$$
$$\Theta^1(b_{n-k,k})=(n+1-k)b_{n-k,k}-ka_{n+1-k,k-1},$$
$$k=1,2,\dots,n-1,n,\quad \Theta^1(b_{n,0})=(n+1)b_{n+1}.$$

Finally we obtain the expression of the differential operator (21) in system's (1) coefficients:

$$\Theta^1=\sum_{k=0}^{n-1}\left[(k+1)a_{n-k,k}-(n-k)b_{n-k-1,k+1}\right]\frac{\partial}{\partial a_{n-k,k}}+(n+1)a_{0,n}\frac{\partial}{\partial a_{0,n}}+$$
$$(n+1)b_{n,0}\frac{\partial}{\partial b_{n,0}}+\sum_{k=1}^{n}\left[(n+1-k)b_{n-k,k}-ka_{n+1-k,k-1}\right]\frac{\partial}{\partial b_{n-k,k}}. \tag{23}$$

Takes place

Theorem 1. *The coefficients t_k $(k=0,1,2,\dots,n)$ of the equation (16) are GL-invariants of the degree $2n$ with respect to the system (1) coefficients and with the weight equal to n^2-n such that*

$$t_0=\mu_n,\quad kt_k=\Theta^1(t_{k-1})-(n+1)(n+k-2)t_{k-1}. \tag{24}$$

Proof. From Remark 5 and (17) it follows that each coefficient t_k, $k=0,1,2,\dots,n$, is a homogeneous and isobaric polynomial of variables f_i and g_i (which are called irrational invariants). According to the results of invariant theory (see [3],[4]) every isobaric and homogeneous polynomial of the invariants f_i and g_i will be an invariant of the binary form $F_{n+1}(x,y)$. Because $F_{n+1}(x,y)$ is a comitant of the system (1) it results that each coefficient t_k is a GL-invariant of the system (1).

We express the operator (21) in the terms of f_i and g_i. Because $\Theta^1(u_i)=u_i$ and $\Theta^1(v_i)=v_i$ we easily obtain that $\Theta^1(d_{i,j})=2d_{i,j}$ and $\Theta^1(f_i)=2nf_i$. Now we shall prove that $\Theta^1(g_i)=(n-1)g_i+f_i$.

Let $u_i\neq 0$. From conditions (10), (12) and (14) we obtain

$$\Theta(g_i)=\sum_{k=1}^{n}(-1)^{k+1}\Theta(C_n^k\, b_{n-k,k}v_i^{n-k}u_i^{k-1})+\Theta(u_1\dots u_{i-1}u_{i+1}\dots u_{n+1}v_i^n)=$$

$$\sum_{k=1}^{n}(-1)^{k+1}[(n+1-k)(C_n^kb_{n-k,k}-C_n^{k-1}a_{n+1-k,k-1})v_i^{n-k}u_i^{k-1}+$$

$$+(n-1)\sum_{k=1}^{n}(-1)^{k+1}C_n^k\, b_{n-k,k}v_i^{n-k}u_i^{k-1}+2nu_1\dots u_{i-1}u_{i+1}\dots u_{n+1}v_i^n=$$

$$(n-1)g_i+(n+1)u_1\dots u_{i-1}u_{i+1}\dots u_{n+1}v_i^n+\sum_{k=1}^{n}\frac{(-1)^k}{k!}\,(n+1-k)\Omega_{vu}^k(b_{n,0})v_i^{n-k}u^k=$$

$$(n-1)g_i + \frac{1}{u_i} \cdot \frac{\partial F_{n+1}}{\partial x}(v_i, -u_i) = (n-1)g_i + f_i.$$

If $u_i = 0$, then $v_i \neq 0$ and from (12) $g_i = nb_{n-1,1}v_i^{n-1} + u_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i^n$. From (6) it results that $d_{i,j} = -v_i u_j$, $f_i = u_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i^n$ and $\Omega^1_{vu}(u_1 u_2 \dots u_{n+1}) = u_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i$. So, for $\Theta^1(g_i)$ we obtain

$$\Theta^1(g_i) = n(C_n^1 b_{n-1,1} - a_{n,0})v_i^{n-1} + (n-1)C_n^1 b_{n-1,1}v_i^{n-1} + 2nu_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i^n =$$

$$(n-1)g_i - nu_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i^n + (n+1)u_1 \dots u_{i-1}u_{i+1} \dots u_{n+1}v_i^n = (n-1)g_i + f_i.$$

So, the formula for $\Theta^1(g_i)$ is proved. Thus, the operator (21) can be written

$$\Theta^1 = \sum_{i=1}^{n+1} \Big\{ [(n-1)g_i + f_i]\frac{\partial}{\partial g_i} + 2nf_i\frac{\partial}{\partial f_i} \Big\}. \tag{25}$$

We show the recurrence (24) by induction. Let $t_0 = \mu_n = g_1 g_2 \cdot \ldots \cdot g_n g_{n+1}$. By using the operator (25) we have

$$\Theta^1(t_0) = [(n-1)g_1 + f_1]g_2 \cdot \ldots \cdot g_k \cdot \ldots \cdot g_{n+1} + g_1[(n-1)g_2 + f_2]g_3 \cdot \ldots \cdot g_k \cdot \ldots \cdot g_{n+1} +$$
$$\ldots + g_1 g_2 \cdot \ldots \cdot g_{k-1}[(n-1)g_k + f_k]g_{k+1} \cdot \ldots \cdot g_{n+1} +$$
$$g_1 g_2 \cdot \ldots \cdot g_k \cdot \ldots \cdot g_n[(n-1)g_{n+1} + f_{n+1}] = (n-1)(n+1)t_0 + t_1.$$

So, $t_1 = \Theta^1(t_0) - (n-1)(n+1)t_0$ and the recurrence (24) is true for $k = 1$. Now we suppose that the recurrence (24) is true for every positive integer $k = 1, 2, \dots, m$. We shall prove the relation

$$(m+1)t_{m+1} = \Theta^1(t_m) - (n+1)(n+m-1)t_m. \tag{26}$$

Every term of t_m is the product of m different factors from f and $n+1-m$ different factors from g such that the indexes of all factors of this term form a permutation of $\{1, 2, \dots, n+1\}$, for example $P = f_1 f_2 \cdot \ldots \cdot f_m g_{m+1} g_{m+2} \cdot \ldots \cdot g_{n+1}$. The action of the operator Θ^1 on the selected term generates $2nm + (n-1)(n+1-m) = (n+1)(n+m-1)$ terms equal with P and $n-m$ different terms from t_{m+1}. So, among all the generated terms of t_m there exist exactly $m+1$ equal terms from t_{m+1}. We obtain the equality (26). By the mathematical induction the recurrence (24) is true for all $k = 1, 2, 3, \dots, n$. Theorem 1 is proved.

Proposition 2. *If $D_{n+1} \neq 0$, then the values $\theta_i = g_i/f_i$ are the roots of the equation*

$$t_n\theta^{n+1} - t_n\,\theta^n + t_{n-1}\,\theta^{n-1} - \ldots + (-1)^n\,t_1\,\theta + (-1)^{n+1}\,t_0 = 0. \tag{27}$$

From Viette relations for the equation (27) and Lemma 1 results

Proposition 3. *If $D_{n+1} \neq 0$, then the roots θ_i of the equation (27) satisfy the equality*

$$\sum_{k=1}^{n+1} \theta_k = 1. \tag{28}$$

Lemma 2. *If* $D_{n+1} \neq 0$, *then the system (1) has the first integral of the form*

$$(u_1x+v_1y)^{\theta_1}(u_2x+v_2y)^{\theta_2}\cdot\ldots\cdot(u_nx+v_ny)^{\theta_n}(u_{n+1}x+v_{n+1}y)^{\theta_{n+1}}=c, \qquad (29)$$

where θ_i $(i=1,2,\ldots,n+1)$ *are the roots of the equation (27).*

Proof. Let $D_{n+1} \neq 0$. After substitution $y = xz$ the corresponding to the system (1) differential equation has the form

$$-\frac{dx}{x}=\frac{P_n(1,z)dz}{F_{n+1}(1,z)}.$$

For polynomial $P_n(x,y)$ Lagrange's interpolation formulae is applicable :

$$\begin{aligned}P_n(x,y) &= P_n(v_1,-u_1)\frac{(u_2x+v_2y)(u_3x+v_3y)\ldots(u_{n+1}x+v_{n+1}y)}{(-d_{12})(-d_{13})\ldots(-d_{1,n+1})}\\ &+P_n(v_2,-u_2)\frac{(u_1x+v_1y)(u_3x+v_3y)\ldots(u_{n+1}x+v_{n+1}y)}{(+d_{12})(-d_{23})\ldots(-d_{2,n+1})}+\\ &\ldots+P_n(v_n,-u_n)\frac{(u_1x+v_1y)\ldots(u_{n-1}x+v_{n-1}y)(u_{n+1}x+v_{n+1}y)}{(d_{1,n})(d_{2,n})\ldots(d_{n-1,n})(-d_{n,n+1})}\\ &+P_n(v_{n+1},-u_{n+1})\frac{(u_1x+v_1y)\ldots(u_{n-1}x+v_{n-1}y)(u_nx+v_ny)}{(d_{1,n+1})(d_{2,n+1})\ldots(d_{n-1,n+1})(d_{n,n+1})}.\end{aligned}$$

From the last relation the polynomial $P_n(1,z)$ has the following representation

$$P_n(1,z)=\frac{g_1}{f_1}\frac{\partial F_{n+1}}{\partial X_1}+\frac{g_2}{f_2}\frac{\partial F_{n+1}}{\partial X_2}+\ldots+\frac{g_n}{f_n}\frac{\partial F_{n+1}}{\partial X_n}+\frac{g_{n+1}}{f_{n+1}}\frac{\partial F_{n+1}}{\partial X_{n+1}}.$$

Using the equality $\partial X_i(1,z)/\partial z = v_i$ and the factorization (4) of the polynomial $F_{n+1}(1,z)$ we obtain the following differential equation

$$-\frac{dx}{x}=\Big[\frac{g_1}{f_1}\frac{v_1}{u_1+v_1z}+\frac{g_2}{f_2}\frac{v_2}{u_2+v_2z}+\ldots+\frac{g_n}{f_n}\frac{v_n}{u_n+v_nz}+\frac{g_{n+1}}{f_{n+1}}\frac{v_{n+1}}{u_{n+1}+v_{n+1}z}\Big]dz.$$

After integration by using Proposition 3 we obtain the first integral (29). Lemma 2 is proved.

3 The center problem for the system (1)

Let $n = 2m+1$, $m \in \mathbb{N}$ and suppose that $u_i \in \mathbb{C}\setminus\mathbb{R}$ or $v_i \in \mathbb{C}\setminus\mathbb{R}$ for every $i = 1,2,\ldots,2m+1,2m+2$. From [6] the singular point $(0,0)$ of the system (1) is a center if and only if the following condition

$$\int_0^{2\pi}\frac{G_{2m+2}(\cos\alpha,\sin\alpha)}{F_{2m+2}(\cos\alpha,\sin\alpha)}d\alpha=0 \iff \int_0^{2\pi}\frac{T_{2m}(\cos\alpha,\sin\alpha)}{F_{2m+2}(\cos\alpha,\sin\alpha)}\,d\alpha=0 \qquad (30)$$

holds. For each invariant line $X_i = 0$ determined by the equation $F_{2m+2}(x, y) = 0$ we denote by r_i the residue of the rational function $T_{2m}(x, y)/F_{2m+2}(x, y)$:

$$r_i \;=\; \mathrm{res}_{X_i=0}\, \frac{T_{2m}(x, y)}{F_{2m+2}(x, y)}.$$

The following lemma holds:

Lemma 3. *If the homogeneous equation* $F_{2m+2}(x, y) = 0$ *has no nontrivial real solutions and the discriminant* $D_{2m+2} \neq 0$, *then for every* $i = 1, 2, \ldots, 2m+2$ *the relation*

$$r_i = \frac{(2m+2)g_i}{f_i} - 1 = (2m+2)\theta_i - 1 \tag{31}$$

holds.

Proof. We will obtain the value of the residue r_i, corresponding to the invariant line $X_i = 0$, by using Lemma 1. Let us consider the following 2 cases:

1. Let $v_i \neq 0$. The substitution $z = \tan\alpha$ in the last integral from (30) implies the relation

$$\int_{-\infty}^{+\infty} \frac{T_{2m}(1, k)}{F_{2m+2}(1, k)}dk = 0.$$

For each root $k_i = -u_i/v_i$ of the equation $F_{2m+2}(1, k) = 0$ the residue of the rational function $T_{2m}(1, k)/F_{2m+2}(1, k)$ is equal to

$$r_i = \frac{T_{2m}(1, k_i)}{(F_{2m+2})'_k(1, k_i)} = \frac{T_{2m}(1, k_i)}{v_i(F_{2m+2})'_{X_i}(1, k_i)} = \frac{T_{2m}(1, -u_i/v_i)}{v_i(F_{2m+2})'_{X_i}(1, -u_i/v_i)} =$$

$$\frac{T_{2m}(v_i, -u_i)}{f_i} = \frac{(2m+2)g_i - f_i}{f_i} = (2m+2)\theta_i - 1.$$

2. Let $u_i \neq 0$. The substitution $z = \cot\alpha$ in the last integral from (30) implies the relation

$$\int_{-\infty}^{+\infty} \frac{T_{2m}(s, 1)}{F_{2m+2}(s, 1)}ds = 0.$$

For each root $s_i = -v_i/u_i$ of the equation $F_{2m+2}(s, 1) = 0$ the residue of the rational function $T_{2m}(s, 1)/F_{2m+2}(s, 1)$ is equal to

$$r_i = \frac{T_{2m}(s_i, 1)}{(F_{2m+2})'_s(s_i, 1)} = \frac{T_{2m}(s_i, 1)}{u_i(F_{2m+2})'_{X_i}(s_i, 1)} = \frac{T_{2m}(-v_i/u_i, 1)}{u_i(F_{2m+2})'_{X_i}(-v_i/u_i, 1)} =$$

$$\frac{T_{2m}(v_i, -u_i)}{f_i} = \frac{(2m+2)g_i - f_i}{f_i} = (2m+2)\theta_i - 1.$$

Lemma 3 is proved.

If we put $\theta = (r+1)/(n+1)$ in equation (27) then we obtain an equation of degree $n+1$, called the residual equation.

Proposition 4. *If* $D_{n+1} \neq 0$, *then the values* $r_i = (n+1)\theta_i - 1$ *are the roots of the equation*

$$X(r) = c_0\, r^{n+1} + c_2\, r^{n-1} + \ldots + c_n\, r + c_{n+1} = 0, \tag{32}$$

where

$$c_k = \sum_{m=0}^{k} (-1)^m\, (n+1)^m\, C_{n+1-m}^{k-m}\, t_{n+1-m}, \quad (\forall)\ k = 0, 1, \ldots, n, n+1. \tag{33}$$

Remark 7. *The equalities* $t_{n+1} = t_n = (-1)^{n(n+1)/2} D_{n+1}$ *imply the equality* $c_1 = 0$.

The discriminant of the equation (32) has the form

$$R_{n+1} = \text{Res}\,(X(r), X'(r)) = D_{n+1}^{2n} \Delta^2,$$

where

$$\Delta^2 = \prod_{1 \le i < j \le n+1} (r_j - r_i)^2$$

is a GL-invariant of the system (1).

Let us consider that the equation (32) has no real solutions and let $r_{i_1}, r_{i_2}, \ldots, r_{i_{m+1}}$ be the solutions with positive coefficients of the imaginary part. In this case it is known that

$$\int_{-\infty}^{+\infty} \frac{T_{2m}(1,k)}{F_{2m+2}(1,k)} dk = 2\,\pi\, i\, (r_{i_1} + r_{i_2} + \ldots + r_{i_{m+1}}).$$

We construct the polynomial of minimal degree $W(r_1, r_2, \ldots, r_{2m+2})$ such that it is simmetric with respect to variables r_i and has the form

$$W(r_1, r_2, \ldots, r_{2m+2}) = \prod (r_{i_1} + r_{i_2} + \ldots + r_{i_{m+1}}).$$

According to the theorem of the symmetric polynomials there exists some polynomial Φ such that the polynomial W can be expressed through the elementary symmetric polynomials of the variables r_i:

$$W(r_1, r_2, \ldots, r_{2m+2}) = \Phi\Big(\frac{c_2}{c_0}, \frac{c_3}{c_0}, \ldots, \frac{c_n}{c_0}, \frac{c_{n+1}}{c_0}\Big).$$

So, there exists positive integer l such that $V = c_0^l \Phi(\frac{c_2}{c_0}, \frac{c_3}{c_0}, \ldots, \frac{c_n}{c_0}, \frac{c_{n+1}}{c_0})$ is a polynomial of the variables $c_0, c_2, c_3, \ldots, c_{n+1}$.

Takes place

Proposition 5. *The system (1) with imaginary invariant straight lines has a center iff* $V = 0$ *and the residual equation (32) has no real solutions.*

Example 1. *For* $n = 3$ *the system (1) with imaginary invariant straight lines has a center iff at least one of the following two series of conditions is fulfilled:*

(i) $V = c_3 = 0$ *and the inequalities* $c_0 c_2 < 0,\ c_2^2 - 4c_0 c_4 > 0$ *are not fulfilled simultaneously;*

(ii) $V = c_3 = 0,\ c_2^2 - 4c_0 c_4 = 0,\ c_0 c_2 > 0.$

Example 2. *For $n = 5$ the system (1) with imaginary invariant straight lines has a center iff* $V = -c_0c_5^2 + 4c_0c_4c_6 - c_3^2c_4 + c_2c_3c_5 = 0$ *and the residual equation (32) has no real solutions.*

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