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Solution of the center problem for cubic systems with a bundle of three invariant straight lines*

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Abstract. For cubic differential system with three invariant straight lines which pass through the same point it is proved that a singular point with purely imaginary eigenvalues (weak focus) is a center if and only if the focal values g_{2j+1}, $j=\overline{1,5}$, vanish.

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1 Introduction

A cubic system with a singular point with pure imaginary eigenvalues ($\lambda_1 = \overline{\lambda}_2 = i$, $i^2=-1$) by a nondegenerate transformation of variable and time rescaling can be brought to the form

$$\begin{aligned} \tfrac{dx}{dt} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \tfrac{dy}{dt} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv -Q(x,y). \end{aligned} \tag{1}$$

The variables x, y and coefficients $a, b, \ldots, r, s$ in (1) are assumed to be real. A singular point $(0,0)$ is a center or a focus for (1). The problem arises of distinguishing between a center and a focus, i.e. of finding the coefficient conditions on (1) under which $(0,0)$ is, for example, a center. These conditions are called the conditions for a center existence or the center conditions and the problem - the problem of the center.

Note that the singular point $(0,0)$ of the differential system (1) is called also weak focus (fine focus).

It is well known that the origin is a center for (1) if and only if all focal values g_{2j+1}, $j=\overline{1,\ \infty}$, vanish. The focal values are polynomials in coefficients of system (1). For example, the first of them looks as follows

$$g_3 = ac - bd + 2ag - 2bf + cf - dg - 3k + 3l - p + q. \tag{2}$$

If all the g_{2j+1} are zero up to $g_{2\tau+1}$, i.e. $g_{2j+1}=0$, $j=\overline{1,\tau-1}$, and $g_{2\tau+1}\neq 0$, then τ is called the order of the weak focus $(0,0)$.

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It is known also that the system of differential equations (1) has a center at $O(0,0)$ if and only if it has in some neighbourhood of the origin an independent of t holomorphic first integral $F(x,y)=C$ (an holomorphic integrating factor $\mu(x,y)$).

The problem of the center was solved for quadratic system ($k=l=m=n=p=q=r=s=0$) by H. Dulac [10], and for symmetric cubic system ($a=b=c=d=f=g=0$) by K.S. Sibirski [16].

If the cubic system (1) contains both quadratic and cubic nonlinearities, the problem of the center is solved only in some particular cases (see, for example, [2, 4, 6–9, 11–14]).

The quadratic system and symmetric cubic system with a singular point of center type are Darboux integrable, i.e. these systems have a first integral (integrating factor) of the form of product of invariant algebraic curves. Hence, the interest arose to study the center problem for polynomial differential systems with algebraic invariant curves. The problem of integrability for polynomial systems with invariant algebraic curves and, in particular, with invariant straight lines was considered in works [3, 5–8, 17, 20].

The straight line $C+Ax+By=0$ is said to be invariant for (1) if there exists a polynomial $K(x,y)$ such that the identity holds

$$A\cdot P(x,y)-B\cdot Q(x,y)\equiv(C+Ax+By)K(x,y). \tag{3}$$

$K(x,y)$ is called the cofactor of the invariant straight line.

By [6] the cubic system (1) can not have more than four nonhomogeneous invariant straight lines, i.e. straight lines of the form

$$1+Ax+By=0\ (|A|+|B|\neq 0). \tag{4}$$

As homogeneous straight lines $Ax+By=0$ this system can have only the lines $x\pm iy=0$, $i^2=-1$. Hence, the cubic system (1) can not have more than six invariant straight lines. This case is realized. To solve the problem of the center in the case of system (1) with four nonhomogeneous invariant straight lines, it is enough to require the vanishing of the first focal value (Liapunov quantity) g_3 [6]. The vanishing of the first focal value in the case of system (1) with four invariant straight lines among which are also homogeneous ones is not enough for the existence of a center. Also the vanishing of the second focal value g_5 is necessary.

Thus, the cubic system (1) with four invariant straight lines (real, complex, real and complex) has at the origin a singular point of a center type if and only if the first two focal values vanish [7].

If (1) has three invariant straight lines two of which are homogeneous, then the presence of a center at $(0,0)$ is guaranteed by vanishing of the focal values $g_{2j+1}=0$, $j=\overline{1,7}$ [19].

In this paper we study the center problem assuming that the cubic system (1) has three invariant straight lines which pass through the same point.

2 Conditions for the existence of a bundle of tree invariant straight lines

From (3) it results that (4) is an invariant straight line of (1) if and only if A and B are the solutions of the system

$$\begin{aligned} &F_1(A,B) = AB^2 - fAB + bB^2 + rA - lB = 0, \\ &F_2(A,B) = A^2B + aA^2 - gAB - kA + sB = 0, \\ &F_3(A,B) = B^3 - 2A^2B + fA^2 + (c-b)AB - dB^2 - pA + nB = 0, \\ &F_4(A,B) = A^3 - 2AB^2 - cA^2 + (d-a)AB + gB^2 + mA - qB = 0. \end{aligned} \tag{5}$$

The cofactor of (4) is

$$\begin{aligned} K(x,y) \quad = \quad & -Bx + Ay + (aA - gB + AB)x^2 + \\ & (cA - dB + B^2 - A^2)xy + (fA - bB - AB)y^2. \end{aligned} \tag{6}$$

Further, we shall assume that the cubic system (1) has three invariant straight lines which pass through the same point (x_0, y_0). By a rotation and rescaling coordinate axes we can make that $x_0 = 0$, $y_0 = 1$. Consequently, the equation of each invariant straight line of the bundle has the form

$$1 + Ax - y = 0. \tag{7}$$

It is evident that the point $(0,1)$ of the intersection of these straight lines is a singular point for (1), i.e. $P(0,1) = Q(0,1) = 0$. These equalities give $r = -f - 1$, $l = -b$. Substituting $B = -1$, $r = -f - 1$ and $l = -b$ in (5) we find that

$$\begin{aligned} &F_1 \equiv 0,\ F_2 = (a-1)A^2 + (g-k)A - s = 0, \\ &F_3 = (f+2)A^2 + (b-c-p)A - d - n - 1 = 0, \\ &F_4 = A^3 - cA^2 + (a - d + m - 2)A + g + q = 0. \end{aligned}$$

From the above equalities we can see that the system (1) can have three distinct invariant straight lines of the form (7) iff the following conditions holds:

$$a = 1,\ f = -2,\ k = g,\ l = -b,\ n = -d-1,\ p = b - c,\ r = 1,\ s = 0, \tag{8}$$

$$\begin{aligned} &4(g+q)c^3 + (d-m+1)^2c^2 + 18(d-m+1)(g+q)c + 4d^3 \\ &-12(m-1)d^2 + 12(m-1)^2d - 27(g+q)^2 - 4(m-1)^3 \neq 0. \end{aligned} \tag{9}$$

In the conditions (8),(9) the straight line (7) is invariant for (1) iff A satisfies the equation

$$A^3 - cA^2 + (m-d-1)A + g + q = 0. \tag{10}$$

The left-hand side of the inequality (9) coincides with the discriminant of the equation (10) and (9) gives that the roots A_1, A_2, A_3 of the (10) are not equal: $A_i \neq A_j\ \forall i \neq j$.

Using (5),(8) and (9) it is easy to show that along with three invariant straight lines of the form (4) the system (1) has also one more invariant nonhomogeneous straight line if and only if at least one of the following two series of conditions holds:

$$\begin{array}{l} a = r = 1,\ b = l = s = 0,\ f = -2,\ k = g, \\ n = -d - 1,\ p = -c,\ q = g(d+1), \end{array} \tag{11}$$

$(d+1)(d+2) \neq 0$. The straight line $1 + (d+1)y = 0$;

$$\begin{array}{l} a = r = 1,\ f = -2,\ k = g,\ l = -b,\ n = -d-1,\ p = b - c,\ s = 0, \\ (m - gc + g^2)(b+g)^2 - (dg - q)(b+g) + bg = 0, \\ 2(b+g)^3 - (b+c)(b+g)^2 - (d+2)(b+g) + b = 0, \end{array} \tag{12}$$

$bg(b+g) \neq 0$. The straight line $1 + gx - g(b+g)^{-1}y = 0$.

3 Sufficient center conditions

a) Darboux integrability.

Lemma 1. *The conditions (11) are sufficient for the origin to be a center for the system (1).*

Proof. Assume that $(d+1)(d+2) \neq 0$ and that the inequality (9) holds.

Denote by $A_1,\ A_2, A_3$ the roots of the equation (10). Then

$$c = A_1 + A_2 + A_3,\ m = A_1A_2 + A_1A_3 + A_2A_3 + d + 1,\ g = -A_1A_2A_3/(d+2).$$

The straight lines $l_j \equiv 1 + A_jx - y = 0,\ j = 1,2,3,$ of the bundle and the straight line $l_4 \equiv 1 + (d+1)y = 0$ have, respectively, the cofactors (see (6)):

$$\begin{array}{rcl} K_1(x,y) & = & x + A_1y - A_1A_2A_3(d+2)^{-1}x^2 + \\ & & (1 + d + A_1A_2 + A_1A_3)xy - A_1y^2, \\ K_2(x,y) & = & x + A_2y - A_1A_2A_3(d+2)^{-1}x^2 + \\ & & (1 + d + A_1A_2 + A_2A_3)xy - A_2y^2, \\ K_3(x,y) & = & x + A_3y - A_1A_2A_3(d+2)^{-1}x^2 + \\ & & (1 + d + A_1A_3 + A_2A_3)xy - A_3y^2, \\ K_4(x,y) & = & x(d+1)(y - 1 + A_1A_2A_3(d+2)^{-1}x). \end{array} \tag{13}$$

The system (1) has the first integral of the form $l_1^{\alpha_1}l_2^{\alpha_2}l_3^{\alpha_3}l_4^{\alpha_4} = const$, where $\alpha_j,\ j = \overline{1,4},\ \sum|\alpha_j| \neq 0$ are generally complex numbers if and only if the following identity holds

$$\sum_{j=1}^{4} \alpha_jK_j(x,y) \equiv 0. \tag{14}$$

Substituting (13) in (14) we obtain

$$\begin{array}{l} \alpha_1 = (A_2 - A_3)(A_2A_3 + d + 2), \\ \alpha_2 = (A_3 - A_1)(A_1A_3 + d + 2), \\ \alpha_3 = (A_1 - A_2)(A_1A_2 + d + 2), \\ \alpha_4 = (A_1 - A_2)(A_1 - A_3)(A_2 - A_3)/(d+1). \end{array}$$

Therefore, in conditions (11), (9), $(d+1)(d+2) \neq 0$, the system (1) has in some neighborhood of the origin a holomorphic first integral of the form $F(x,y) = const$ and this means that $(0,0)$ is a center of (1).

Since the center variety is closed in the space of coefficients of the system (1), then $(0,0)$ will be a center also in the cases when one or both of the inequalities $(d+1)(d+2) \neq 0$ and (9) do not hold.

Lemma 2. *The conditions*

$$\begin{array}{l} a = r = 1, f = -2, k = g, l = -b, n = -d-1, p = b-c, \\ q = g + d(b+g),\ s = 0,\ (b+g)^4 - (2b+c)(b+g)^3 \\ +(b^2+bc+m+1)(b+g)^2 + bd(b+g) - b^2 = 0, \\ 2(b+g)^3 - (b+c)(b+g)^2 - d(b+g) - b - 2g = 0. \end{array} \tag{15}$$

are sufficient for the origin to be a center for system (1).

Proof. In the conditions (8) the equality $g_3 = 0$ (see (2)) looks $d(b+g)+g-q = 0$, from where we express $q:\ q = g+d(b+g)$. Note that the conditions (15) are included in (12) if in the last we put $q = g + d(b+g)$.

Assume that the inequalities (9) and $bg(b+g)(1+(b+g)(A_2+A_3)-A_2A_3) \neq 0$, where A_2, A_3 are the roots of the equations (10), hold. Denote $\nu = b+g$. The last two equalities from (15) give us

$$d = 2\nu^2 - (b+c)\nu - 2 + b\nu^{-1},\ m = c\nu - \nu^2 - 1 + 2b\nu^{-1}.$$

In this case the equation (10) looks

$$(A-\nu)(A^2 - (c-\nu)A - 2\nu^2 + (b+c)\nu + b\nu^{-1} = 0.$$

We put $A_1 = \nu$ and let A_2, A_3 be the roots of the quadratic equation $A^2 - (c-\nu)A - 2\nu^2 + (b+c)\nu + b\nu^{-1} = 0$. Then

$$b = \nu(A_2-\nu)(A_3-\nu)/(\nu^2+1),\ c = A_2 + A_3 + \nu.$$

The invariant straight lines

$$l_j = 1 + A_jx - y,\ j = \overline{1,3},\ l_4 = 1 + \nu^2 + (1 + A_2\nu + A_3\nu - A_2A_3)(\nu x - y)$$

of the system (1) have, respectively, the cofactors:

$$\begin{array}{rcl} K_1(x,y) & = & x + \nu y + \big[(\nu(-A_2A_3 + A_2\nu + A_3\nu + 1))x^2 \\ & & +((A_2+A_3)\nu^3 + (1-A_2A_3)\nu^2 - (A_2+A_3)\nu + A_2A_3 - 1)xy \\ & & +(\nu(-\nu A_2 - \nu A_3 + A_2A_3 - 1))y^2\big]/(\nu^2+1), \\ K_2(x,y) & = & x + A_2y + \big[(\nu(A_2\nu + A_3\nu - A_2A_3 + 1))x^2 \\ & & +(\nu^3A_2 + \nu^2 - \nu A_2 - 2\nu A_3 + 2A_2A_3 - 1)xy \\ & & +(\nu^3 - 2\nu^2A_2 - \nu^2A_3 + \nu A_2A_3 - A_2)y^2\big]/(\nu^2+1), \\ K_3(x,y) & = & x + A_3y + \big[(\nu(A_2\nu + A_3\nu - A_2A_3 + 1))x^2 \end{array}$$

$$
\begin{aligned}
&+(\nu^3 A_3+\nu^2-\nu A_3-2\nu A_2+2A_2A_3-1)xy\\
&+(\nu^3-2\nu^2A_3-\nu^2A_2+\nu A_2A_3-A_3)y^2\big]/(\nu^2+1),\\
K_4(x,y) \quad = \quad &(1+(A_2+A_3)\nu-A_2A_3)(\nu x-y+1)(\nu y+x)/(\nu^2+1).
\end{aligned}
$$

The system (1) has an integrating factor of the Darboux form $\mu(x,y) = l_1^{\beta_1} l_2^{\beta_2} l_3^{\beta_3} l_4^{\beta_4}$ (this means that $(0,0)$ is a center) if and only if the numbers β_1, β_2, β_3, β_4 satisfy the identity

$$\sum_{j=1}^{4} \beta_j K_j(x,y) \equiv \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}.$$

Substituting in this identity the expressions of the cofactors and identifying the coefficients of x, y, x^2, xy and y^2, we obtain that

$$
\begin{aligned}
\beta_1 &= 1,\\
\beta_2 &= (A_2A_3\nu - A_2 + 2A_3)/(A_2-A_3),\\
\beta_3 &= (A_2A_3\nu + 2A_2 - A_3)/(A_3-A_2),\\
\beta_4 &= (A_2A_3\nu^2 - A_2A_3 + 2\nu A_2 + 2\nu A_3 + 2)/(A_2A_3 - \nu A_2 - \nu A_3 - 1).
\end{aligned}
$$

Lemma 3. *The conditions*

$$
\begin{gathered}
a=-n=r=1,\ d=s=0,\ f=-2,\ k=q=g,\\
l=-b,\ p=b-c,\ m=3+(b+g)(3c-3b-5g)
\end{gathered}
$$

are sufficient for the origin to be a center for the system (1).

Proof. In the conditions of lemma 3 the equation (10) looks

$$A^3-(2b+\beta)A^2+(8b\nu+3\nu\beta-5\nu^2+2)A-2b+2\nu=0, \tag{16}$$

where $\nu=b+g$, $\beta=c-2b$. Suposse that (16) has three diferent roots A_1, A_2, A_3. The straight line $l_j \equiv 1+A_jx-y=0$ of the bundle has the cofactor $K_j(x,y) = x+A_jy+(\nu-b)x^2+(1-A_j^2+2bA_j+\beta A_j)xy+(b-A_j)y^2$ $(j=1,2,3)$.

The identity

$$\sum_{j=1}^{3} \beta_j K_j(x,y) \equiv \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}$$

holds if

$$
\begin{aligned}
\beta_1 \quad = \quad &(-2A_2A_3+\beta A_2+\beta A_3-2b\beta+16b\nu-\beta^2+\\
&6\beta\nu-10\nu^2+6)/((A_1-A_2)(A_1-A_3)),\\
\beta_2 \quad = \quad &(2A_1A_3-\beta A_1-\beta A_3+2b\beta-16b\nu+\beta^2-\\
&6\beta\nu+10\nu^2-6)/((A_1-A_2)(A_1-A_3)),\\
\beta_3 \quad = \quad &(-2A_1A_2+\beta A_1+\beta A_2-2b\beta+16b\nu-\beta^2+\\
&6\beta\nu-10\nu^2+6)/((A_1-A_2)(A_1-A_3)).
\end{aligned}
$$

Therefore, $\mu(x,y)=l_1^{\beta_1}l_2^{\beta_2}l_3^{\beta_3}$ is an integrating factor of the system (1) and, consequently, $(0,0)$ is a center.

By the closedness of the center variety in the space of coefficients of (1) the singular point $(0,0)$ will be the center type also in the cases when the equation (16) has multiple roots.

b) Symmetry.

Let

$$\begin{aligned}
&a=r=1,\ c=6b+5g,\ f=-2,\ k=g,\ l=-b,\ p=-5(b+g),\ s=0,\\
&d=-5(b+g)(4+(3b+2g)(4b+3g))/(13b+10g),\\
&m=(5(b+g)(3b+2g)(5b+4g)-b-10g)/(13b+10g),\\
&n=(5(b+g)(3b+2g)(4b+3g)+7b+10g)/(13b+10g),\\
&q=-(5(b+g)^2(3b+2g)(4b+3g)+20b^2+27bg+10g^2)/(13b+10g).
\end{aligned}\tag{17}$$

The system (1) with (17), after the change of coordinates

$$X=\frac{x}{1-y},\quad Z=\frac{y}{1-y},$$

defines the following equation of nonlinear oscillations:

$$P_4(X)ZZ'=-XP_0(X)-3XP_1(X)Z-P_2(X)Z^2-P_3(X)Z^3,\tag{18}$$

where

$$\begin{aligned}
P_0(X)=\ &1+gX,\\
P_1(X)=\ &(19b+10g-5(4b+3g)(3b+2g)(b+g)-(20b^2+bg-\\
&10g^2+5(4b+3g)(3b+2g)(b+g)^2)X)/(3(13b+10g)),\\
P_2(X)=\ &(13b^2+10bg+(6b-5(4b+3g)(3b+2g)(b+g))X-\\
&(20b^2+14bg+5(4b+3g)(3b+2g)(b+g)^2)X^2)/(13b+10g),\\
P_3(X)=\ &b,\\
P_4(X)=\ &(13b+10g+(13b+10g)(6b+5g)X+\\
&(6b+5(9b+7g)(3b+2g)(b+g))X^2+\\
&(20b^2+14bg+5(4b+3g)(3b+2g)(b+g)^2)X^3)/(13b+10g).
\end{aligned}$$

The substitution $Z=\frac{P_0(X)Y}{1-P_1(X)Y}$ [15] reduces the equation (18) to the form

$$Q_4(X)YY'=-X-Q_2(X)Y^2-Q_3(X)Y^3,$$

where

$$\begin{aligned}
Q_2(X)\ &\equiv\ P_0(X)P_2(X)-3XP_1^2(X)+P_0'(X)P_4(X),\\
Q_3(X)\ &\equiv\ 2XP_1^3(X)-P_0(X)P_1(X)P_2(X)+P_0^2(X)P_3(X)+\\
&\quad\ P_0(X)P_1'(X)P_4(X)-P_0'(X)P_1(X)P_4(X),\\
Q_4(X)\ &\equiv\ P_0(X)P_4(X).
\end{aligned}$$

By Theorem 9.4 of [1] in the case $Q_3(X) = X^{2j+1}\widetilde{P}(X)$, $\widetilde{P}(0) \neq 0$ the origin is a center for the equation (18) if and only if the system of equations

$$\begin{aligned} &y^4R^3(x)Q_3^5(y) - x^4R^3(y)Q_3^5(x) = 0,\\ &xQ(x)R^2(y) - yQ(y)R^2(x) = 0, \end{aligned} \tag{19}$$

where

$$R(X) \equiv Q_4(X)[Q_3(X) - XQ_3'(X)] + 3XQ_2(X)Q_3(X),$$
$$Q(X) \equiv Q_4(X)[R'(X)Q_3(X) - 3R(X)Q_3'(X)] + 4Q_2(X)Q_3(X)R(X)$$

has in some neighborhood of $X = 0$ a holomorphic solution

$$Y = \phi(X), \quad \phi(0) = 0,\ \phi'(0) = -1. \tag{20}$$

Let us consider the following two series of conditions on the coefficients of system (1):

$$\begin{aligned} &a = r = 1,\ c = g = k = -3b/2,\ d = -5,\ f = -2,\\ &l = -b,\ m = -7,\ n = 4,\ p = 5b/2,\ q = b; \end{aligned} \tag{21}$$

$$\begin{aligned} &a = r = 1,\ c = 6b + 5g,\ f = -2,\ k = g,\ p = -5(b + g),\\ &l = -b,\ d = -5(b + g)(4 + (3b + 2g)(4b + 3g))/(13b + 10g),\\ &m = (5b(b + g)(3b + 2g) - 21b + 30g)/(13b + 10g),\\ &n = (5b(b + g)(3b + 2g) - 8b + 40g)/(13b + 10g),\\ &q = -4g,\ r = 1,\ s = 0,\ (3b + 2g)(b + g)^2 + b - 2g = 0. \end{aligned} \tag{22}$$

Remark. *The conditions (21) (respectively, (22)) can be obtained from conditions (17) if to the last we add the equality* $g = -3b/2$ *(respectively,* $(3b + 2g)(b + g)^2 + b - 2g = 0$*).*

Lemma 4. *Each of conditions (21), (22) are sufficient conditions for the system (1) to have a center at the origin.*

Proof. Assume first that the conditions (21) hold. Then the equalities (19) have a solution in the form of (20):

$$Y = \frac{3b^2X^2 - 20bX + 12 + (bX - 2)\sqrt{3(3b^2X^2 - 20bX + 12)}}{2b(2 - 3bX)}.$$

Now, assume that the conditions (22) hold. From $(3b + 2g)(b + g)^2 + b - 2g = 0$ we find that

$$b = 2\nu(1 - \nu^2)/(\nu^2 + 3),\ g = \nu(1 + 3\nu^2)/(\nu^2 + 3),$$

where ν is a parameter. The conditions (22) look:

$$\begin{aligned} &a = r = 1,\ b = 2\nu(1 - \nu^2)/(\nu^2 + 3),\ c = \nu(3\nu^2 + 17)/(\nu^2 + 3),\\ &d = -5(3\nu^2 + 1)/(\nu^2 + 3),\ f = -2,\ g = \nu(1 + r\nu^2)/(\nu^2 + 3),\\ &k = \nu(1 + r\nu^2)/(\nu^2 + 3),\ l = 2\nu(\nu^2 - 1)/(\nu^2 + 3),\\ &m = (13\nu^2 - 1)/(\nu^2 + 3),\ n = 2(7\nu^2 + 1)/(\nu^2 + 3),\\ &p = -5\nu,\ q = -4\nu(3\nu^2 + 1)/(\nu^2 + 3),\ s = 0. \end{aligned}$$

Finally, it is easy to verify that ecuations (19) have a solution in the form of (20):

$$Y = -\frac{3\nu^2X^2 + 10\nu X + 3 - (\nu X + 1)\sqrt{3(3\nu^2X^2 + 10\nu X + 3)}}{2(3\nu X + 1)}.$$

4 The problem of the center

In this section by "$\Longrightarrow$" we will understand "further it is used".

Theorem. *The order of a weak focus for cubic differential systems with a bundle of three invariant straight lines is at most five.*

Proof. Without loss of generality, we shall consider the cubic system (1) with conditions (8),(9). In the same conditions we shall calculate the first five focal values using the algorithms described in ([18]). The first one looks: $g_3 = q - g - d(b + g)$ (see (2), (8)). From $g_3 = 0$ we find q:

$$q = g + d(b + g)$$

and substitute into the expression for g_5. We have $g_5 = bd(m - (b + g)(3c - 3b - 5g) - 2d - 3)$. If $b = 0$ then $\Longrightarrow$ Lemma 1, if $d = 0$ then $\Longrightarrow$ Lemma 3.

Let

$$bd \neq 0 \tag{23}$$

and

$$m = (b + g)(3c - 3b - 5g) + 2d + 3.$$

The third focal value being cancelled by non-zero factors is of the form $g_7 = f_1f_2$, where

$$f_1 = 2(b + g)^3 - (b + c)(b + g)^2 - d(b + g) - b - 2g,$$
$$f_2 = 6b + 5g - c.$$

If $f_1 = 0 \Longrightarrow$ Lemma 2. Further, we shall consider that $bdf_1 \neq 0$. Simplify the focal values g_9 and g_{11} by bdf_1.

From $f_2 = 0$ we express c:

$$c = 6b + 5g$$

and substitute it in g_9, g_{11}. The g_9 looks as

$$g_9 = (13b + 10g)d + 5(b + g)(4 + (3b + 2g)(4b + 3g)). \tag{24}$$

If the coefficient d in g_9 is equal to zero, i.e. $g = -13b/10$, then $g_9 = -3b(b^2 + 100)/50 \neq 0$ (see (23)). We require that $13b + 10g \neq 0$. From $g_9 = 0$ (see (24)) express d :

$$d = -5(b + g)(4 + (3b + 2g)(4b + 3g))/(13b + 10g) \tag{25}$$

and substitute it in g_{11}. For g_{11}, after corresponding simplifications, i.e. after elimination of a denominator and non-zero factors, including numerical one, we have

$$g_{11} = (b + g)(3b + 2g)((3b + 2g)(b + g)^2 + b - 2g).$$

If $b+g=0$, then from (25) $d=0$. That is in contradiction with assumption (23). If $(3b+2g)((3b+2g)(b+g)^2+b-2g)=0 \Longrightarrow$ Lemma 4 (in the case of $3b+2g=0$ we have the series (21) of conditions on the coefficients of the system (1) and in the case $(3b+2g)(b+g)^2+b-2g=0$, respectively, the series (22)).

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