BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(41), 2003, Pages 83–90
ISSN 1024–7696

CMC–surfaces, φ–geodesics and the Carathéodory conjecture

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Abstract. A short proof of the Caratheodory conjecture about index of an isolated umbilic on the convex 2–dimensional sphere is suggested.

Mathematics subject classification: 30F30, 53A07, 58F10.
Keywords and phrases: quadratic differential, φ–metric, umbilical point.

1 Introduction

The constant mean curvature (CMC–) surfaces in E^3 are known to admit a continuous family of local, non–trivial, isometric deformations preserving mean curvature of the surface (H–deformations). In the case when surface is compact Umehara [11] showed that the converse is also true.

The maximal and minimal curvature lines of CMC–surface form an orthogonal net which is called *réseau de Bonnet*, cf Cartan [3]. The Bonnet Theorem says that if the CMC–surface is simply connected and umbilic–free, then under H–deformations the orthogonal net "rotates" through a constant angle which can be taken as a parameter of deformation.

If the CMC–surface is not simply connected or umbilic–free, Cartan seems to be the first to ask about possible scenario of evolution of *réseau de Bonnet* under the H–deformations.

In the present note we study [1] evolution of the orthogonal nets in the case when CMC–surface is simply connected with a single umbilic or, equivalently, doubly connected and umbilic–free. Namely, if we "pinch" the umbilic, the CMC–surface becomes an annulus whose points undergo H–deformations according to the Bonnet Theorem. In general, the rotation angle is no longer constant at all the points because annulus cannot be covered by a single chart.

However, the Bonnet Theorem implies that every curve of the orthogonal net is a φ–geodesic line whatever H–deformations are applied to a CMC–surface. Metric φ is given by the linear element $ds = |\varphi||dz|$, where φdz^2 is a holomorphic quadratic "differential" associated to the CMC–surface. Of course, $\varphi(0) = 0$ at the umbilical point.

This observation is crucial, because the φ–geodesics near n–th order zero of a holomorphic quadratic form are well–understood due to the works of Strebel [10].

[1] For the reasons which will be clear later.

Roughly speaking, the φ–geodesics fill–up the annulus either by "hyperbolas" or "radii". Therefore, possible configurations of *réseau de Bonnet* near the umbilic looks like a singularity with the finite number of hyperbolic and parabolic sectors.

Despite independent interest, the orthogonal nets are auxiliary for us. We postulate different fact here: H–deformations of orthogonal nets give an amazingly simple proof to the *Caratheodory'sche Vermutung (Conjecture)*:

Theorem 1. *Let S^2 be a C^∞ surface which bounds a convex compact body in the Euclidean space E^3. Then S^2 has at least two umbilical points. In other words, the Euler-Poincaré index of isolated umbilical point is at most $+1$.*

(A short overview of this conjecture can be found in [1]; see also [2],[5],[7].)

2 φ–geodesics

Until further indications, M is a simple domain of the complex parameter z. Let us consider the holomorphic functions $\varphi(z)$ vanishing at the unique point of M which we identify with 0. An order $n \geq 1$ is assigned to 0, if there exists a complex constant $a \neq 0$ such that $\varphi(z) = az^n + O(|z|^{n+1})$.

Flat metric φ with the cone singularity of angle $(n+2)\pi$ is given by the formula

$$|ds| = |\varphi||dz|,$$

provided $\varphi(z)dz^2$ is a quadratic form on M. By a *φ–geodesic line* in M one understands the line conisting of the shortest arcs relatively metric φ. Any two points in M (including 0) may be joined by the unique φ–geodesic line. Strebel classified the possible types of φ–geodesics in the neighborhood of n–th order zero by proving the following lemma.

Lemma 2. *([10]) Any two points in a neighborhood M of n–th order zero of holomorphic 2–form $\varphi(z)dz^2$ can be joined by a unique φ–geodesic. Moreover, each φ–geodesic is either an arc defined by the equation $Arg\ \varphi(z)dz^2 = Const$, or is composed of the two radii centered in 0 with the minimal angle $\geq 2\pi/(n+2)$.*

The foliation $\mathcal{F}$ on $M\backslash 0$ is said to be *geodesic* if every leaf of $\mathcal{F}$ is a φ–geodesic line. Before we state the general lemma on the structure of geodesic foliations, let us consider an example when all $\mathcal{F}$'s can be obtained by a "brute force".

If 0 is a double zero, then the φ–metric is given by the linear element $ds^2 = (u^2+v^2)(du^2+dv^2)$ where $u+iv$ is a natural parameter. The metric $|ds|$ is Liouville's and the geodesic lines in this metric are completely integrable. The general integral is known to be of the form

$$\int \frac{du}{\sqrt{u^2-a}} \pm \int \frac{dv}{\sqrt{v^2+a}} = a',$$

where a, a' are two independent constants. Easy calculations show that no information will be lost if we suppose $a = 0$. The integral takes the form $\ln|u| \pm \ln|v| = a'$. The geodesic foliation is described by two "families of curves": $v = Cu$ and $v = C/u$, where C is an arbitrary constant. Thus, $\mathcal{F}$ near a double zero is either the "node" with the geodesics radii tending to 0, or the "saddle" with four sectors filled–up by the geodesic "hyperbolas".

Let w be a finite "word" on the alphabet consisting of two symbols h and p. We introduce the elementary operations on w:

(i) a cyclic permutation of the symbols in w, and

(ii) a contraction of the p–symbol: $p^2 = p$.

Two words are *equivalent* $w_1 \sim w_2$ if and only if w_2 can be obtained from w_1 by the elementary operations. The equivalence class of word w is denoted by $[w]$.

Fix an integer number $n \geq 1$. To every symbol h in w we assign a *weight* $|h| = 2\pi/(n+2)$. To every symbol p we assign the weight $|p| = \alpha_i$, where α_i is a positive real. The weight of w is an additive function equal to the sum of weights of the symbols entering w. The equivalence class $[w]$ is called *normalized* if $|w| = 2\pi$ for all $w \in [w]$. (Note that the weight of w is one and the same for all $w \in [w]$.)

Lemma 3. *Let h and p stay for the hyperbolic and the parabolic sectors of the singularity w, respectively. We encode the singular point w by a sequence of symbols h and p in the order the h– and the p–sectors occur when turning clockwise around the singularity. Then:*

(i) each φ–geodesic foliation $\mathcal{F}$ is topologically equivalent to the singularity w of a normalized equivalence class $[w]$;

(ii) each normalized equivalence class $[w]$ can be realized as a φ–geodesic foliation $\mathcal{F}$ with the singularity $w \in [w]$ in a neighborhood of n–th order zero of φ for some $n \geq 1$.

Proof. Denote by M a neighborhood of the n–th order zero of φ. Let us introduce a partial order for the points $x, y \in M$: $x \leq y$ if and only if $Arg\ x \leq Arg\ y$. If $x \in M$ is an arbitrary point, then by Lemma 2 the φ–geodesic line through x is either (i) the hyperbola $Arg\ \varphi dz^2 =\ Const$ or (ii) the radius Ox. Let us consider the first possibility.

(i) The hyperbola $Arg\ \varphi dz^2 =\ Const$ must tend to the asymptotic rays Oz_1, Oz_2 with $z_1 < x < z_2$, enclosing the angle $2\pi/(n+2)$. Clearly, the only possibility to the geodesic foliation $\mathcal{F}$ is to form a hyperbolic sector $z_1 O z_2$. Of course, along Oz_1 and Oz_2 $Arg\ \varphi dz^2$ is constant.

(ii) Let Ox be the geodesic radius through x, distinct from the boundary radii of the hyperbolic sector. Then through the nearby points $|x-y| < \varepsilon$ one can draw the geodesic radii Oy's. Denote by $y_1 O y_2$ the maximal connected parabolic sector filled-up with the geodesic radii. Clearly, $y_1 < x < y_2$. The angle enclosed between two boundary radii, we denote by α. In general, $0 \leq \alpha \leq 2\pi$.

If the hyperbolic sector h is followed by another hyperbolic sector h, we write this as hh. If h is followed by a parabolic sector, we put it as hp. A parabolic sector p followed by the parabolic sector p, gives a larger parabolic sector $p = pp$ and the contraction rule (ii) follows. Of course, the "weights" of the sectors are equal to the angles swept by the sectors.

Finally, according to the definition of normalized equivalence class, each singularity consists of sequence of parabolic and hyperbolic sectors; every curve in these sectors is a geodesic arc.

The part (ii) of Lemma 3 is proved by the similar argument.

3 CMC–surfaces

Every smooth immersion $f : M \to E^3$ of an orientable surface M into the Euclidean space E^3 induces a Riemann structure on M; let $z = u + iv$ be the corresponding local parameter. With respect to z the first fundamental form can be written as $ds^2 = e^{2\lambda}|dz|^2$.

If $ldu^2 + 2mdudv + ndv^2$ is the second fundamental form, we consider a complex quadratic form φdz^2, such that $\varphi(z) = \frac{1}{2}(l - n) - im$. The Mainardi–Codazzi equations imply that φ is holomorphic on M if and only if $f(M)$ is a CMC–surface. Locally, along the lines of minimal and maximal curvature $Arg\ \varphi dz^2 = 0$ and $\varphi(0) = 0$ at the umbilic points.

A *continuous deformation* f_t of the immersion $f = f_0$ is the isometry of surface M such that $M \times [0, 1] \to E^3$ is a continuous mapping. The continuous deformation f_t is called an *H–deformation* if $H_t = H$ for all $t \in [0, 1]$, where $H : M \to \mathbb{R}$ is the mean curvature function.

The CMC–surfaces are known to admit a non–trivial H–deformations and in the case of compact surfaces, they are the only ones with such a property. Of course, there are known many examples of compact CMC–surfaces of genus $g > 0$.

What happens with the lines of principal (i.e. minimal or maximal) curvature of the CMC–surface during an H–deformation? If M is a local CMC–surface without umbilics, the principal curvature lines of $f_0(M)$ and $f_t(M)$ form two families of the parallel lines intersecting each other with the constant angle proportional to the parameter t (the Bonnet Theorem, see e.g.[4]). Note that if we fix the φ–metric on M corresponding to $f_0(M)$, then the principal curvature lines of $f_t(M)$ coincide with the φ–geodesic lines of the inclination t. If the umbilical points are allowed, then a law is given by the following lemma.

Lemma 4. *Suppose that $M_0 = f_0(M)$ is a canonical CMC–surface with the quadratic function $\varphi = z^n, n \geq 1$. Let φ be a metric on M corresponding to M_0. If $M_t = f_t(M)$ is an H–deformation of M_0, then one of the two principal curvature lines of M_t coincide with the φ–geodesic lines on M for any $t \geq 0$.*

Proof. In the polar coordinate system the coefficients of the second fundamental form of surface M_t are given by the equations:

$$\begin{aligned} l &= He^{\lambda} + |z|^n \cos(2t - n\ Arg\ z), \\ m &= |z|^n \sin(2t - n\ Arg\ z), \\ n &= He^{\lambda} - |z|^n \cos(2t - n\ Arg\ z), \end{aligned} \tag{1}$$

where t is a parameter of the H–deformation, cf [11]. The following two cases are possible.

(i) An H–deformation, such that t is constant on M. It can be immediately seen that in new coordinates $\tilde{u} = \cos t\ u + \sin t\ v$, $\tilde{v} = -\sin t\ u + \cos t\ v$ the first and the second forms of surfaces M_0 and M_t are the same. By the fundamental theorem, surfaces M_0 and M_t may differ only by a rigid motion in E^3. Thus, the H–deformation is trivial.

(ii) A non–trivial H–deformation. By item (i), t varies for the points of M. Thus far, associated to every $z \in M\backslash 0$, there is a chart in which the second fundamental form of surface $M_t(z)$ writes as

$$l = He^{\lambda} + \cos 2t, \quad m = \sin 2t, \quad n = He^{\lambda} - \cos 2t,$$

where t is the deformation parameter, cf [13]. A straightforward calculation shows that the principal curvature lines of the surface $M_t(z)$ coincide with the φ–geodesic lines of the slope t on M. (This fact follows also from the Bonnet Theorem.) Since every regular point $z \in M$ can be endowed with such a chart, Lemma 4 is proved.

4 Proof of Theorem 1

Take a convex C^∞ immersion $f_0 : S^2 \to E^3$ of the 2–sphere into the Euclidean space E^3 which is not totally umbilic (i.e. there are no $U \subseteq S^2$ such that $f_0(U)$ is a part of the round sphere). In other words, umbilics are supposed isolated and their number is finite. Denote by ds_0 a Riemann metric on S^2 induced by the immersion f_0 and by $H : S^2 \to \mathbb{R}$ the corresponding mean curvature function.

Definition 1. *By a Hopf spheroid in E^3 we understand a convex C^∞ immersion $f : S^2 \to E^3$ such that there exists at least one umbilical point p and a small closed disc $D \ni p$ such that $H(D) = \ Const.$*

Lemma 5. *There exist infinitely many Hopf spheroids in E^3.*

Proof. By the results of Wente and Kapouleas any compact orientable surface S_g of genus $g > 0$ admits an immersion into E^3 which is a CMC-surface with $H > 0$; cf.[6],[12]. Fix $g \geq 2$ and consider the lines of principal curvature of any such immersion. By the index argument, there exists an umbilic $p \in S_g$ and a small closed disc $D \ni p$ which is a convex local surface in E^3. We separate this local surface from S_g. To obtain a Hopf spheroid, it remains to complete this piece of CMC-surface to a C^∞ immersion $S^2 \to E^3$. By Urysohn's lemma this can be done in an infinite number of ways. □

Lemma 6. *For the Hopf spheroids the Caratheodory conjecture is true.*

Proof. Without loss of generality we can assume that the umbilic point p of Hopf spheroid is unique. (For otherwise, if there are more than one umbilic then we are done.) Since a Hopf spheroid is locally CMC, we apply Lemma 4 to identify the curvature lines in the disc $D \ni p$ with φ-geodesic lines in the vicinity of a singularity w.

Let $w \in [w]$ be a word of the minimal length in the normalized equivalence class $[w]$. According to Lemma 3, there exists a singularity of order n whose topological type is encoded by the sequence w of symbols h and p. Let w admit $\langle h \rangle$ symbols of type h and $\langle p \rangle$ symbols of type p. By the normalization axiom, $\langle h \rangle \leq n + 2$.

To estimate the Euler–Poincaré index of singularity w, note that the parabolic sectors make no contribution to the index value and the number $\langle p \rangle$ can be neglected. To the contrary, if there are no hyperbolic sectors (i.e. $w = p$) we necessarily have one parabolic sector. The general formula is true:

$$Ind\ w = \begin{cases} 1 - \frac{\langle h \rangle}{2} & \text{if} \quad w \neq p, \\ +1 & \text{if} \quad w = p. \end{cases}$$

In either case $Ind\ w \leq 1$ and by the index argument the conjecture follows.

Now we are ready to finish the proof of Theorem 1. But first we wish to outline the main idea. To every convex C^∞ immersion $f_0 : S^2 \to E^3$ one can relate a Hopf spheroid. This spheroid is uniquely defined by f_0 and is a 'modification' of f_0 which has an interesting 'mechanical' interpretation.

Suppose that f_0 is a convex steel ball filled-up with a gas under a pressure. Let p be an isolated umbilic of f_0. We drill a small hole in p and glue-up a soap film D into this hole maintaining a pressure [2] inside the ball. We also 'deform' slightly the 'edges' of the cut in order to keep the modified surface $f : S^2 \to E^3$ in the class C^∞. We claim that f is a Hopf spheroid.

Indeed, $f(D)$ is a local CMC-surface with an umbilic point $p \in D$. Moreover, the index of umbilic on the Hopf spheroid is equal to the index of p on f_0. (This is because the foliation by principal curvature lines at the 'steel part' of ball remains intact.) In general, if $\mathcal{F}_0$ and $\mathcal{F}$ are foliations by the principal curvature lines on f_0 and f, respectively, then $\mathcal{F}$ is obtained from $\mathcal{F}_0$ by a *homotopy of opening of a leaf*; cf [9].

Let f_0 be as above. If p is an isolated umbilic of f_0 then we take a closed disc $|D| \leq r$ centred at the point p. We are going to define a local CMC-surface $f(D)$. Let $z = u + iv$ be a local parameter which corresponds to a part of CMC-surface with an umbilic; see the beginning of this section. By the results of Umehara [11]

[2] The absolute value of the pressure depends on how 'flat' is the surface at the point p. Of course, by 'pressure' we understand difference of pressures inside and outside the steel ball.

(see also [3],[4]) there exists a family of isometric H-deformations depending on a real parameter t:

$$I = e^{2\lambda}|dz|^2, \qquad II_t = l du^2 + 2m\; dudv + n dv^2, \tag{2}$$

with l, m and n given by equations (1). The Mainardi-Codazzi and Gauss equations for I, II_t:

$$\frac{\partial\varphi}{\overline{\partial z}} = \frac{\partial H}{\partial z}, \qquad |\varphi|^2 = e^{4\lambda}(H^2 - K), \tag{3}$$

where $\varphi = e^{it}z^n$ is a complex quadratic form φdz^2, are satisfied for any real t. (Indeed, the first equation is true since $H = \; Const$ and φ is holomorphic; the second equation follows from $|\varphi e^{it}| = |\varphi|$ and the fact that H-deformation is an isometry.) Therefore, the fundamental forms (2) are realized by a concrete local CMC-surface for each real number t.

Let $f_t(D)$ be a family of local CMC-surfaces described above. Denote by A an annular region which surrounds disc D:

$$A = \{z = u + iv|\; r \le |z| \le r + \varepsilon\}. \tag{4}$$

To glue-up $f_t(D)$ properly, we fix the metric λ so that $\lambda|_{\partial A_{r+\varepsilon}} = \lambda|_{\partial A_r}$, where the left part denotes a metric on the exterior boundary of A which is induced by metric of the surface f_0. The boundary condition $\lambda|_{\partial A_r}$ gives a unique solution $f_{t=t^*}(D)$ to the Gauss equation, so that a representative in the family $f_t(D)$ is fixed.

To obtain a C^∞ Hopf spheroid it remains to conjugate $f_{t^*}(D)$ with the rest of the sphere:

$$f(S^2) = \begin{cases} f_{t^*}(D), & \text{if} \quad z \in D \subset Int\; D_{r+\varepsilon}, \\ f_0(S^2), & \text{if} \quad z \in S^2 \backslash D_{r+\varepsilon}. \end{cases} \tag{5}$$

By the Urysohn Lemma, function f in formula (5) can be chosen C^∞ for an arbitrary small ε, see formula (4). Moreover, taking r sufficiently small we can fix number n (see (1)) equal to the order of quadratic form φ at point p of the surface f_0. (Such an order is correctly defined for any φ, not necessary holomorphic.)

Thus, the surface f given by equation (5) is a Hopf spheroid. By the Lawson-Tribuzy theorem f is uniquely defined up to a rigid motion in E^3; see [8]. To finish the proof of Caratheodory conjecture, it remains to notice that passage from f_0 to f gives us a homotopy $h(\mathcal{F}_0) = \mathcal{F}$ between foliations induced by curvature lines. In particular, $Ind\; p_0 = Ind\; p$. By Lemma 6, the Caratheodory conjecture follows. □

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Received September 5, 2002