BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(41), 2003, Pages 78–82
ISSN 1024–7696

Note on multiple zeta–values*

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Abstract. We introduce some generating functions $g(t;x)$ for multiple zeta values. They satisfy linear differential equations $Pg + x^a g = 0$ of the Fuch type. We find WKB-type expansions for g as $x \to \infty$. M41

1 Certain familiar generating function

D. Zagier had presented in [8] an 'ultra–simple' Calabi's proof of the Euler formula $\zeta(2) = \pi^2/6$. That proof uses the integral $\int_0^1 \int_0^1 (1-xy)^{-1}$ (equal to $\frac{3}{4}\zeta(2)$) and the substitution $(x,y) = \left(\frac{\sin u}{\cos v}, \frac{\sin v}{\cos u}\right)$. Below I present a proof which is even more simple (in my opinion).

The function $f_2(x) = \frac{\sin \pi x}{\pi x}$ has the Taylor expansion

$$f_2 = 1 - \frac{\pi^2}{3!}x^2 + \frac{\pi^4}{5!}x^4 - \frac{\pi^6}{7!}x^6 + \dots \tag{1}$$

and the infinite product representation

$$f_2 = \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\dots. \tag{2}$$

Comparing the coefficients of x^2 we see immediately that $\sum \frac{1}{n^2} = \frac{\pi^2}{3!}$.

We recall that the *multiple ζ–values* are defined as follows:

$$\zeta(a_1, \dots, a_k) = \sum_{0<n_1<\dots<n_k} \frac{1}{n_1^{a_1} \dots n_k^{a_k}} \tag{3}$$

for integer $a_i \geq 1$, $a_k \geq 2$ (see [8]).

Therefore f_2 is the generating function for multiple zeta–values,

$$f_2(x) = 1 - \zeta(2)x^2 + \zeta(2,2)x^4 - \zeta(2,2,2)x^6 + \dots. \tag{4}$$

Since any $\zeta(2,\dots,2)$ (k arguments) is expressed via $\zeta(2l)$'s for $l \leq k$, one finds that

$$\zeta(2k) = \pi^{2k} \times \textit{rational number}.$$

For example, $\zeta(2,2) = \frac{1}{2}\sum_{m \neq n} m^{-2}n^{-2} = \frac{1}{2}\left(\sum_{m,n} - \sum_{m=n}\right) m^{-2}n^{-2}$, that gives $\zeta(4) = \zeta(2)^2 - 2\zeta(2,2) = \pi^4/36 - 2\pi^4/120 = \pi^4/90$; similarly, one finds $\zeta(6) = 3\zeta(2,2,2) + \frac{3}{2}\zeta(2)\zeta(4) - \frac{1}{2}\zeta(2)^3 = \pi^6/945$, etc.

Note also that instead of $\frac{\sin \pi x}{\pi x}$ one could use $\cos \pi x$ as a generating function for some quantities easily expressed via the multiple zeta–values.

*Supported by Polish KBN Grant No 2 P03A 010 22

2 Irrationality of $\zeta(2)$

This result was firstly proved by A. Legendre [5]. The proof we present below is a modification of the proof of irrationality of π given in the book of A. Shidlovskiĭ [7].

One begins with the identities

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \varphi(y)\cos y &= \varphi(y)\sin y\,|_{-\pi/2}^{\pi/2} - \int \varphi'(y)\sin y \\
&= [\varphi(\frac{\pi}{2}) + \varphi(-\frac{\pi}{2})] - \int \varphi''(y)\cos y \qquad (5)\\
&= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \Big[\varphi(\frac{\pi}{2}) + \varphi(-\frac{\pi}{2})\Big] - \Big[\varphi''(\frac{\pi}{2}) + \varphi''(-\frac{\pi}{2})\Big] + \ldots
\end{aligned}$$

Suppose that $\zeta(2)$ is rational, i.e. that $\frac{\pi^2}{4} = \frac{a}{b}$, $a, b \in \mathbb{Z}$. We take $\varphi(y) = \frac{b^n}{n!}(\frac{\pi^2}{4} - y^2)^n = \frac{(a-by^2)^n}{n!}$ in (5). The left-most (positive) integral in (5) behaves like $C^n/n!$ for large n and takes values between 0 and 1. Next, for $k < n$ we have $\varphi^{(k)}(\pm\pi/2) = 0$ and for $k = 2l \geq n$ and even, the polynomial $\varphi^{(k)}(y)$ is a sum of terms $\frac{1}{n!}b^m\left(y^{2m}\right)^{(2l)} \times$ integer $= \frac{(2l)!}{n!}\binom{2m}{2l}b^m y^{2(m-l)} \times$ integer . Thus the right-most combination in (5) should represent an integer number (a contradiction).

Note that this proof relies essentially upon the fact that $(\cos x)'' = -\cos x$, which follows from the 'functional equation' $\cos(\pi \pm y) = -\cos y$.

The proof of transcendency of $\zeta(2)$ was firstly given by F. Lindemann [6]. It is more complicated, so we do not present it here.

3 Other generating functions

Analogously to (3) one can define the functions

$$\begin{aligned}
f_{a_1,\ldots,a_k}(x) &= 1 - \zeta(a_1,\ldots,a_k)x^a + \zeta(a_1,\ldots,a_k,a_1,\ldots,a_k)x^{2a} \\
&\quad -\zeta(a_1,\ldots,a_k,a_1,\ldots,a_k,a_1,\ldots,a_k)x^{3a} + \ldots,
\end{aligned}$$

$a = a_1 + \ldots + a_k$.

It turns out that this function can be represented as $g(x;t)|_{t=1}$, where the function $g = g_{a_1,\ldots,a_k}(t;x)$ satisfies the following linear differential equation

$$Pg + x^a g = 0. \qquad (6)$$

Here $P = RQ^{a_1-1}RQ^{a_2-1}\ldots RQ^{a_1-1}$ is a differential operator defined via $Q = (1-t)\partial$, $R = t\partial$ and $\partial = \partial/\partial t$. Moreover $g(x;t)$ is analytic near $t = 0$ and $g(x,t) = 1 + O(t)$.

To see this, following [8], introduce the functions

$$I(\varepsilon_1,\ldots,\varepsilon_m;t) = \int\cdots\int_{0<t_1<\ldots<t_m<t} \frac{dt_1}{A_{\varepsilon_1}(t_1)}\cdots\frac{dt_m}{A_{\varepsilon_m}(t_m)}$$

(indexed by $\varepsilon_1 = 0,1,\ \varepsilon_2 = 0,1,\ldots,\varepsilon_m = 0,1$) with

$$A_0(t) = t,\ A_1(t) = 1-t.$$

Next, define

$$\tilde{\zeta}(a_1,\ldots,a_k;t) = I(\underbrace{1,0,\ldots,0}_{a_1},\ldots,\underbrace{1,0,\ldots,0}_{a_k};t);$$

one finds that $\zeta(a_1,\ldots,a_k) = \tilde{\zeta}(a_1,\ldots,a_k;1)$ (see [8]).

If $\mathbf{1}$ denotes the constant function $\mathbf{1}(t) \equiv 1$, then one has the formula

$$\begin{aligned}\tilde{\zeta}(a_1,\ldots,a_k;\cdot) &= \left[\partial^{-1}t^{-1}\right]^{a_k-1}\partial^{-1}(1-t)^{-1}\ldots\left[\partial^{-1}t^{-1}\right]^{a_1-1}\partial^{-1}(1-t)^{-1}\mathbf{1}\\ &= P^{-1}\mathbf{1}.\end{aligned}$$

Therefore the function

$$g = 1 - \tilde{\zeta}(a_1,\ldots,a_k;t)x^a + \tilde{\zeta}(a_1,\ldots,a_k,a_1,\ldots,a_k;t)x^{2a} - \ldots \tag{7}$$

equals

$$[(I - x^aP^{-1} + x^{2a}P^{-2} - \ldots)\mathbf{1}](t) = \left[(I + x^aP^{-1})^{-1}\mathbf{1}\right](t).$$

It implies that g satisfies the equation $(I + x^aP^{-1})g \equiv 1$ and, in consequence, the equation (6).

Example 1. In the case $k = 1$ and $a_1 = 2$ the equation (6) becomes the hypergeometric equation

$$(1-t)\partial(t\partial g) + x^2 g = 0$$

(with singular points at $t = 0,1,\infty$). Its characteristic exponents (i.e. the powers α in the solutions $(t-t_0)^\alpha + \ldots$ as $t \to t_0$ or $t^\alpha + \ldots$ as $t \to \infty$) are the following: $\lambda = \lambda' = 0$ at $t = 0$; $\rho = 0, \rho' = 1$ at $t = 1$; $\tau = x, \tau' = -x$ at $t = \infty$. It follows (see [1]) that our distinguished solution is the hypergeometric function

$$g_2(x;t) = F(x,-x;1;t).$$

In [4] one can find the following interesting identities (proved by Broadhurst):

$$g_{1,3}(\sqrt{2}x;t) \equiv F(x,-x;1;t)F(ix,-ix;1;t), \quad f_{1,3}(\sqrt{2}x) = f_4(x).$$

Generally, the equation (6) is of the Fuchs type (i.e. with regular growth of solutions at singular points). Its characteristic equations (for the characteristic exponents) are the following: $\alpha^{a_k}(\alpha-1)^{a_{k-1}}\ldots(\alpha-k+1)^{a_1}$ at $t = 0$; $\alpha(\alpha-1)\ldots(\alpha-a+k)\cdot(\alpha-a_k+1)(\alpha-a_k-a_{k-1}+2)\ldots(\alpha-a_k-\ldots-a_2+k-1)$ at $t = 1$ and $(-1)^k\alpha^a + x^a = 0$ at $t = \infty$.

This implies that the monodromy operators $\mathcal{M}_0$ and $\mathcal{M}_1$, induced by analytic prolongation of solutions to (6) along simple loops surrounding $t = 0$ and $t = 1$, are unipotent (with eigenvalues equal to 1). (Maybe this explains the fact that the multiple zeta–values generate the 'ring of periods of the pro-nilpotent completion of $\pi_1(\mathbb{C}P^1\backslash\{0,1,\infty\})$', see [3,8]). The monodromy operator $\mathcal{M}_\infty$ associated with a loop around $t=\infty$ is diagonalizable with different eigenvalues $e^{-2\pi i\alpha}$, $(-1)^k\alpha^a+x^a=0$.

The series in (7) defines g in the disc $|t|<1$, but $g(x;\cdot)$ can be prolonged to a multi-valued holomorphic function with ramifications at $t=1$ and $t=\infty$; (the further branches of g ramify also at $t=0$). Near $t=1$ one has the representation $g=h_0(t-1)+h_1(t-1)\log(t-1)+\ldots+h_r(t-1)\log^r(t-1)$ with analytic $h_j(z)$ near $z=0$. Note that $\zeta(a_1,\ldots,a_k)=h_0(0)$.

We refer the reader to the (very algebraic) paper of A. Goncharov [3] for further results about multiple zeta–values.

4 Asymptotic as $x\to\infty$

The equation (6) for large parameter x is solved using the WKB method. This means that one represents a solution as a finite sum of terms of the form

$$e^{xS(t)}[\varphi_\gamma(t)x^\gamma+\varphi_{\gamma-1}(t)x^{\gamma-1}+\ldots].$$

The 'action' S satisfies the 'Hamilton–Jacobi equation'

$$t^{a-k}(1-t)^k\left(S'\right)^a+1=0, \tag{8}$$

the coefficient φ_γ satisfies the 'transport equation' of the form

$$\varphi_\gamma'+W(t)\varphi_\gamma=0 \tag{9}$$

(with some rational function W) and the other coefficients $\varphi_{\gamma-m}$ satisfy some non-homogeneous equations (whose homogeneous parts are like in (9) and the rests depend on $S',S'',\ldots,\varphi_\gamma,\ldots,\varphi_{\gamma-m+1}$).

The Hamilton-Jacobi equation (8) has solutions of the form of Schwarz–Christoffel integral

$$S(t)=S_j(t)=\xi_j\cdot\int_0^t\tau^{k/a-1}(1-\tau)^{-k/a}d\tau, \tag{10}$$

where ξ_j is a root of (-1) of order a. The transport equation (9) is solved as follows:

$$\varphi_\gamma(t)=\varphi_{\gamma,j}(t)=C_j\cdot t^\mu(1-t)^\nu$$

for some exponents μ,ν depending on the situation. By the initial condition $g(x;0)=1$ the first exponent γ and the constants C_j must be chosen after expanding $e^{xS_j(t)}$ at $t=0$ and solving some further transport equations (we shall not do it). For the same reason the initial limit in the integral (10) is equal to 0.

From this the following expansion formula for the generating function $f_{a_1,\ldots,a_k}=g_{a_1,\ldots,a_k}(x;1)$ follows:

$$f_{a_1,\ldots,a_k}\sim\sum_{j=1}^a e^{\beta\xi_jx}\left[\varphi_{\delta,j}(1)x^\delta+\varphi_{\delta-1,j}(1)x^{\delta-1}+\ldots\right], \tag{11}$$

where $\beta = B(\frac{k}{a}, 1-\frac{k}{a}) = \frac{\pi}{\sin \pi k/a}$ and the constants $\varphi_{\eta,j}(1)$, $\eta \leq \delta$ are (theoretically) calculable. In general, one cannot expect convergence in (11).

It seems that this method would give some insight into the nature of the coefficients of the generating functions $f_{a_1,\ldots,a_k}$.

Example 2. Consider the function $g_3(x;t)$. One finds that $\xi_1 = -1$, $\xi_{2,3} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, $\beta = \frac{2\pi}{\sqrt{3}}$ and $\varphi_{\gamma,j}(t) = C_j \cdot t^{-1/3}(1-t)^{2/3}$. This suggests that the zeta–numbers $\zeta(3), \zeta(3,3), \zeta(3,3,3), \zeta(9), \zeta(15), \ldots$ have something common with the numbers π, i and $\sqrt{3}$. Maybe this is the way to show the transcendency of $\zeta(3)$. (Recall that the irrationality of $\zeta(3)$ was shown by R. Apéry [2], see also [4]).

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Received December 6, 2002