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The Lyapunov stability in restricted problems of cosmic dynamics

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Abstract. Majority of cosmic dynamical problems are described by Hamiltonian systems. In this case the Lyapunov stability problem is the toughest problem of qualitative theory, but for two freedom degrees KAM–theory (Kolmogorov–Arnold–Moser methods) allows for the complete study [1–3]. For application of Arnold–Moser theorem [4] it is necessary to make finite sequence of Poincaré–Birkhoff canonical transformations [5] for Hamiltonian normalization. With the help of Symbolic System "Mathematica" [6] we determine the conditions of Lyapunov stability and instability of equilibrium points of restricted n–body problems [7].

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1 Introduction

Let have the $2n$–dimensional Hamiltonian system

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}\ , \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}\ , \tag{1}$$

where the Hamiltonian $H(p,q)$ is of the type

$$H(p,q) \equiv H_0(p) + \mu H_1(p,q)\ , 0 \le \mu < 1\ ,$$

where its perturbate part H_1 fulfils the condition

$$H_1(p,q) \equiv H_1(p, q + (2\pi)).$$

In addition we assume $H(p,q)$ to be 2π–periodical on $q_1, q_2, ..., q_n$ and analytical on $2n$–dimensional symplectic manifold

$$G_{2n} = \{p \in G_n, \|Imq\| < \rho < 1, \|Imq\| = \sum_{s=1}^{n} \mid Imq_s \mid\}\ ,$$

where G_n denotes a n–dimensional torus manifold in euclidean space. The variables (p,q) usually are referred to as "action – angle" coordinates [8]. The system of differential equations (1) describes the models of cosmic dynamics with the potential gravitational fields. The general and restricted newtonian n–body problems belong to this type.

According to H. Poincaré [9], it is necessary to do a full analytical and qualitative investigation of the system (1).

The problem of integration of the system (1) consists in finding a nondegenerate canonical mapping $G_{2n} \to G^*_{2n}, (p,q) \to (P,Q)$, that reduces the system (1) to the following one:

$$\frac{dP}{dt} = 0, \qquad \frac{dQ}{dt} = \frac{\partial H^*}{\partial P} .$$

It follows from this that in manifold G^*_{2n} one has

$$H^*(P) \equiv H(p,q).$$

On the base of this problem it is necessary to find effective methods of constructing periodical and quasi–periodical solution families of (1), and the investigation of the asymptotic evolution of trajectories of system (1) when $t \to \pm\infty$.

In KAM–theory the transformation $(p,q) \to (P,Q)$ is constructed with the help of an infinite sequence of convergent and nondegenerate canonical substitutions

$$(p,q) \leftrightarrow (p^{(1)},q^{(1)}) \leftrightarrow (p^{(2)},q^{(2)}) \leftrightarrow \ldots \leftrightarrow (p^{(\infty)},q^{(\infty)}) \equiv (P,Q). \tag{2}$$

Convergence of the iterative process (2) is guaranteed by the method of accelerated convergence [10], in which the $k-th$ iteration has μ^{2^k}–order, i.e.

$$(p^{(k)}, \Delta q^{(k)}) = O(\mu^{2^k}),$$

where $\Delta q^{(k)}$ stands for the perturbation of the phase variable $q^{(k)}$.

In the classical methods, the $k-th$ iteration has μ^k–order, which means

$$(p^{(k)}, \Delta q^{(k)}) = O(\mu^k).$$

The process (2), constructed with the use of classical methods for the Hamiltonian systems of the dimensions $4, 6, 8, ...(n \geq 2)$, will be divergent. Therefore, H. Poincaré demonstrated [9] that in classic perturbation theory the sequence $(p,q) \to (P,Q)$ similar to (2) is divergent in G_{2n}.

Manifolds of convergence of canonical transformations (2) represent an infinite sequence of inclusions

$$G_{2n} \supset G^{(1)}_{2n} \supset G^{(2)}_{2n} \supset \ldots \supset G^{(\infty)}_{2n} \equiv G^*_{2n} \ ,$$

where $G^*_{2n} \neq \emptyset$ and

$$G^*_{2n} = \{P \in G^*_n, \|ImQ\| < \rho^* \leq \rho < 1\} \ .$$

V. Arnold demonstrated [11] that, unfortunately, the phase manifolds G^*_n and $\bar{G}_n = G_n \setminus G^*_n$ are everywhere dense in $G_n = G^*_n \cup \bar{G}_n$. C. Siegel has shown in [12], that in G^*_n the following inequality is true

$$|(k, \omega(p))| \geq \frac{K(\omega)}{\|k\|^{n+1}} \ ,$$

where $\omega(p) = \dfrac{\partial H_0}{\partial p}$, and the measures of manifolds $G_n^*, \bar{G_n}$ are

$$mesG_n^* = 1 - \varepsilon, \quad mes\bar{G_n} = \varepsilon << 1 \ .$$

For study of Lyapunov stability problem it is not necessary to construct the infinite sequence (2), but it is sufficient to consider 4–8 iterations for and only for $n = 2$. This fact is the main conclusion from the Arnold–Moser theorem [4].

In fact, if we represent the Hamiltonian $H(p, q)$ in neighbourhood of the equilibrium point $(0, 0, 0, 0)$ in series form, we have

$$H(p, q) = H_2(p, q) + H_3(p, q) + H_4(p, q) + \dots,$$

where $H_k(p, q)$ are homogenous k–degree polynomials in $p = (p_1, p_2), q = (q_1, q_2)$.

The Arnold–Moser theorem's formulation is [4]:

If new (transformed) Hamiltonian has the form

$$W(\psi_1, \psi_2, T_1, T_2) = W_2(T_1, T_2) + W_4(T_1, T_2) + \dots,$$
$$W_2(T_1, T_2) = \sigma_1 T_1 - \sigma_2 T_2, \quad W_4(T_1, T_2) = c_{20} T_1^2 + c_{11} T_1 T_2 + c_{02} T_2^2 \ , \tag{3}$$

and is such that:

1) eigenvalues of linear system

$$\frac{dT_1}{dt} = -\frac{\partial W_2}{\partial \psi_1} = 0, \frac{d\psi_1}{dt} = \frac{\partial W_2}{\partial T_1} = \sigma_1,$$
$$\frac{dT_2}{dt} = -\frac{\partial W_2}{\partial \psi_2} = 0, \frac{d\psi_2}{dt} = \frac{\partial W_2}{dT_2} = -\sigma_2,$$

are the numbers $\pm i\sigma_1, \pm i\sigma_2$;

$$2) \quad n_1\sigma_1 + n_2\sigma_2 \neq 0, \ \textit{for } 0 < |n_1| + |n_2| \leq 4 \ ,$$

and

$$3) \quad c_{20}\sigma_2^2 + c_{11}\sigma_1\sigma_2 + c_{02}\sigma_1^2 \neq 0;$$

then the equilibrium point

$$T_1 = T_2 = \psi_1 = \psi_2 = 0$$

of the Hamiltonian system with the Hamiltonian function W (3) is stable in Lyapunov sense [13].

While analyzing this theorem, we conclude that it is necessary to transform only expressions $H_2(p, q), H_3(p, q), H_4(p, q)$ to new forms $W_2, W_3 = 0, W_4$, in order to study the Lyapunov stability of the equilibrium point

$$p_1 = p_2 = q_1 = q_2 = 0$$

in the "nonresonant case".

2 Determination of equilibrium points

One application of the Arnold – Moser theorem is the study of stability in Lyapunov sense of Lagrange triangle in the famous, restricted circular problem of three bodies [4, 9, 14]. The other one is the study of equilibrium points stability in the restricted circular N–body problem [7, 15].

The differential equations of this dynamical problem in uniformly rotating coordinate system P_0xyz have the form [15]:

$$\begin{aligned} \frac{d^2x}{dt^2} - 2\omega_n \frac{dy}{dt} &= -\frac{m_0 x}{r^3} + \frac{\partial R}{\partial x} \ , \\ \frac{d^2y}{dt^2} + 2\omega_n \frac{dy}{dt} &= -\frac{m_0 y}{r^3} + \frac{\partial R}{\partial y} \ , \\ \frac{d^2z}{dt^2} &= -\frac{m_0 z}{r^3} + \frac{\partial R}{\partial z} \ , \end{aligned} \tag{4}$$

$$R(x,y,z) = \frac{\omega_n^2}{2}(x^2+y^2) + m\sum_{k=1}^{n}\left[\frac{1}{\Delta_k} - \frac{xx_k+yy_k+zz_k}{r_k^3}\right] \ ,$$

$$\Delta_k^2 = (x-x_k)^2 + (y-y_k)^2 + (z-z_k)^2 \ ,$$

$$r^2 = x^2+y^2+z^2, \ r_k^2 = x_k^2+y_k^2+z_k^2, \ k = 1,...,n \ ,$$

$$x_k = a_0\cos\frac{2\pi(k-1)}{n}, \ y_k = a_0\sin\frac{2\pi(k-1)}{n}, \ z_k = 0, \ k = 1,..,n \ ,$$

$$\omega_n = \sqrt{\frac{1}{a_0^3}\left[m_0 + \frac{m}{4}\sum_{k=2}^{n}\left(\sin\frac{\pi(k-1)}{n}\right)^{-1}\right]} \ ,$$

$$n = N-2 \ ,$$

ω_n is the angle speed of coordinate system P_0xyz relative to the original system, and also is the angle speed of regular polygon $P_1P_2...P_n$ in vertexes of which masses $m_1 = m_2 = ... = m_n \neq 0$ are situated round central body P_0 with mass m_0. If $m_0 = 0$ we have Lagrange–Wintner gravitational restricted models [15]. Of course it is always possible to write the equations (4) in the Hamiltonian form.

Determination of equilibrium positions of system (4) comes to solutions of the following nonlinear, functional equation system:

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0 \ ,$$

$$-\frac{m_0 x}{r^3} + \frac{\partial R}{\partial x} = -\frac{m_0 y}{r^3} + \frac{\partial R}{\partial y} = -\frac{m_0 z}{r^3} + \frac{\partial R}{\partial z} = 0 \ ,$$

or

$$\omega_n^2 x - \frac{m_0 x}{r^3} + m\sum_{k=1}^{n}\left[\frac{x_k - x}{\Delta_k^3} - \frac{1}{a_0^2}\cos\frac{2\pi(k-1)}{n}\right] = 0 \ ,$$

$$\omega_n^2 y - \frac{m_0 y}{r^3} + m \sum_{k=1}^{n} \left[\frac{y_k - y}{\Delta_k^3} - \frac{1}{a_0^2} \sin \frac{2\pi(k-1)}{n} \right] = 0 \ , \tag{5}$$

$$- \frac{m_0 z}{r^3} + m \sum_{k=1}^{n} \frac{z}{\Delta_k^3} = 0 \ .$$

In the system (5) the quantities x, y, z are unknowns.

Last equation from (5) for $z = 0$ is always realized. Then all equilibrium points of system (4) are located in the plane P_0xy. It can be shown that for any $n \geq 2$ the system (5) is equivalent to the system [15]:

$$\begin{aligned} &\omega_n^2 x - \frac{m_0 x}{r^3} + m \sum_{k=1}^{n} \frac{x_k - x}{\Delta_k^3} = 0 \ , \\ &\omega_n^2 y - \frac{m_0 y}{r^3} + m \sum_{k=1}^{n} \frac{y_k - y}{\Delta_k^3} = 0 \ . \end{aligned} \tag{6}$$

For the famous restricted 3–body problem ($n = 1$) the equations (5) are of the form

$$\begin{aligned} &\omega_1^2 x - \frac{m_0 x}{r^3} + m \left(\frac{1 - x}{\Delta_1^3} - 1 \right) = 0 \ , \\ &\omega_1^2 y - \frac{m_0 y}{r^3} - \frac{m y}{\Delta_1^3} = 0 \ . \end{aligned} \tag{7}$$

For $y = 0$ the first equation from (7) has three solutions, which have been determined by Euler (collinear solutions). For $y \neq 0$ the system (7) has two solutions, which were determined by Lagrange (two equilateral triangles P_0P_1P). It is known that collinear points are unstable in first approximation for arbitrary values of parameter m.

Research of Lagrange triangle stability has a 200–year history. At first G. Gascheau, E. Routh and A. Lyapunov studied the triangle stability in first approximation [4]. The condition of this stability is

$$0 \leq m < \bar{m} = \frac{9 - \sqrt{69}}{18} = 0.0385209....$$

The stability in Lyapunov sense was studied by H. Poincaré, A. Lyapunov, G. Birkhoff, C. Siegel, V. Arnold, A. Deprit, J. Moser, A. Leontovich, A. Markeev, A. Sokolski, V. Sebehely and ultimate results were achieved on the base of KAM–theory [4, 16].

Using CSS "Mathematica", we have solved the equations (6) and counted the coordinates of equilibrium points in restricted 4, 5, 6, 7 – body problems. We found that all the "radial" [15] points are unstable in first approximation for all values $m \geq 0$, and the "bisectorial" [15] points are stable in first approximation for

$0 \leq m < m^*$, where the parameter m^* for different values of N is represented as follows:

N	$0 \leq m < m^*$
4	$0 \leq m < 0.085...$
5	$0 \leq m < 0.023...$
6	$0 \leq m < 0.0094...$
7	$0 \leq m < 0.0047...$

For all the values $0 \leq m < m^*$ all eigenvalues of the matrix of linear Hamiltonian equations in neighborhood of any bisectorial point S_i are the numbers $\pm\beta i, i = \sqrt{-1}$.

3 Research of Lyapunov stability

In order to use the Arnold–Moser theorem, one has to construct the operation of Birkhoff normalization of Hamiltonians with accuracy up to the fourth degree of local coordinates.

If we translate the origin of the coordinate system from point P_0 to any point S_i with coordinates x^*, y^* with the help of expressions

$$\begin{cases} X = x - x^*, \\ Y = y - y^*, \\ P_X = p_x - p_{x^*}, \\ P_Y = p_y - p_{y^*}, \end{cases}$$

and we pass to canonical variables (X, Y, P_X, P_Y), using classical transformations, we will receive, for example, the Hamiltonian $H(6)$ of the restricted problem of 6 bodies in the form:

$$\begin{aligned} H(6) = &- \left((X+x^*)^2 + (Y+y^*)^2\right)^{-1/2} - m\Bigg(\left((X+x^*)^2 + (Y+y^*-1)^2\right)^{-1/2} + \\ &\left((X+x^*-1)^2 + (Y+y^*)^2\right)^{-1/2} + \left((X+x^*+1)^2 + (Y+y^*)^2\right)^{-1/2} + \\ &\left((X+x^*)^2 + (Y+y^*+1)^2\right)^{-1/2} \Bigg) + \omega_4\Big((Y+y^*)(P_X+p_x^*) - \\ &(X+x^*)(P_Y+p_y^*)\Big) + 1/2\left((P_X+p_x^*)^2 + (P_Y+p_y^*)^2\right). \end{aligned} \tag{8}$$

Obviously Hamiltonian differential equations of restricted 6–body problem in the phase space (X, Y, P_X, P_Y) admit the solution

$$X = Y = P_X = P_Y = 0.$$

The performance of Birkhoff normalization of equations depends on the coordinates of concrete equilibrium point. In what follows, we will consider the bisectorial point S_1, stable in the first order approximation, with coordinates [17]

$$x^* = y^* = 0.709007,$$

calculated for $m = 0.009 < m^*$.

In small neighborhood of the point S_1, the Hamiltonian (8) is representable in the form of a convergent power series,

$$H = H_2(X, Y, P_X, P_Y) + H_3(X, Y) + H_4(X, Y) + ...,$$

where H_k are homogeneous of k–th degree polynomials and

$$\begin{aligned} H_2 &= -0.258702(X^2 + Y^2) + 0.5(P_X^2 + P_Y^2) - 1.44885XY + \omega_4(YP_X - XP_Y), \\ H_3 &= -0.148050513(X^3 + Y^3) + 1.5163341(X^2Y + XY^2), \\ H_4 &= 0.39066344(X^4 + Y^4) - 0.587145981(X^3Y + XY^3) - 3.53151X^2Y^2. \end{aligned} \tag{9}$$

The expressions (9) indicate that the quadratic form $H_2(X, Y, P_X, P_Y)$ contains the term $\omega_4(YP_X - XP_Y)$, which is the first obstacle on the way of Lyapunov stability investigation. Therefore, at first, we perform the nondegenerate canonical transformation

$$(X, Y, P_X, P_Y) \to (q_1, q_2, p_1, p_2),$$

$$[X, Y, P_X, P_Y]^T = A \cdot [q_1, q_2, p_1, p_2]^T, \tag{10}$$

where the matrix A is defined in such a way that the new transformed Hamiltonian $K \quad (H(X, Y, P_X, P_Y) \to K(q_1, q_2, p_1, p_2))$ has the form

$$K(q_1, q_2, p_1, p_2) = K_2(q_1, q_2, p_1, p_2) + K_3(q_1, q_2, p_1, p_2) + K_4(q_1, q_2, p_1, p_2) + ...,$$

and its quadratic form K_2 does not contain the expressions $q_1p_2, q_2p_1, q_2p_2, q_1p_1, p_1p_2, q_1q_2$. The matrix A has the form

$$A = \begin{bmatrix} -2.74006 & 2.32275 & 2.27271 & 2.9743 \\ 0.204828 & 0.173633 & -3.55404 & -3.76981 \\ -1.96543 & 1.6661 & 1.44773 & 2.34872 \\ 0 & 0 & 2.44107 & 2.87964 \end{bmatrix}.$$

The matrix $A = [a_{ij}]$ is symplectic [4]. This means that, in the case of two freedom degrees, it fulfils the symplectic conditions represented by 6 equations:

$$\begin{aligned} a_{11}a_{32} - a_{12}a_{31} + a_{21}a_{42} - a_{22}a_{41} &= 0, \\ a_{11}a_{33} - a_{13}a_{31} + a_{21}a_{43} - a_{23}a_{41} &= 1, \\ a_{11}a_{34} - a_{14}a_{31} + a_{21}a_{44} - a_{24}a_{41} &= 0, \\ a_{12}a_{33} - a_{13}a_{32} + a_{22}a_{43} - a_{23}a_{42} &= 0, \\ a_{12}a_{34} - a_{14}a_{32} + a_{22}a_{44} - a_{24}a_{42} &= 1, \\ a_{13}a_{34} - a_{14}a_{33} + a_{23}a_{44} - a_{24}a_{43} &= 0. \end{aligned}$$

Finding transformation (10) is equivalent to determining the four–by–four matrix A with 16 unknown elements. Solution of the system of homogeneous linear algebraic equations of 16–th order turned out to be possible in practice only with the use of system of symbolic calculations.

Realization of the canonical transformation (10) gives the following expressions for the forms K_2, K_3 and K_4 [17]:

$$K_2 = 0.387142(p_1^2 + q_1^2) - 0.309396(p_2^2 + q_2^2), \tag{11}$$

$$\begin{aligned} K_3 = 20.6018p_1^3 + 17.5615p_2^3 + 5.202q_1^3 - 0.329434q_2^3 \\ -9.8733657q_1^2q_2 + 5.01276q_1q_2^2 + p_1^2(61.2536p_2 + 16.363q_1 \\ -21.60582q_2) + p_2^2(39.3824q_1 - 42.7265q_2) + p_1(58.3744p_2^2 \\ +54.0272p_2q_1 - 45.648q_1^2 - 62.9684p_2q_2 + 81.8039q_1q_2 \\ -35.9368q_2^2) - p_2(51.2223q_1^2 - 90.0176q_1q_2 + 38.754q_2^2)\ , \end{aligned} \tag{12}$$

$$\begin{aligned} K_4 = -73.253p_1^4 - 182.712p_2^4 + 23.3975q_1^4 + 9.51254q_2^4 \\ +p_1^3(-381.659p_2 + 344.902q_1 - 291.661q_2) + p_2^3(-600.158p_1 \\ +470.91q_1 - 377.686q_2) + p_1^2(-725.789p_2^2 + 1163.51p_2q_1 \\ -182.557q_1^2 - 967.849p_2q_2 + 234.407q_1q_2 - 64.4354q_2^2) \\ +p_2^2(1290.8p_1q_1 - 138.024q_1^2 - 1055.12p_1q_2 + 125.044q_1q_2 \\ -4.18566q_2^2) + p_1(-329.286p_2q_1^2 - 82.4767q_1^3 + 375.536p_2q_1q_2 \\ +268.272q_1^2q_2 - 75.8361p_2q_2^2 - 274.302q_1q_2^2 + 89.6987q_2^3) \\ +p_2(-106.674q_1^3 + 332.312q_1^2q_2 - 329.253q_1q_2^2 + 104.971q_2^3) \\ +q_1q_2(-78.8385q_1^2 + 96.5479q_1q_2 - 50.6582q_2^2). \end{aligned} \tag{13}$$

The new variables (q_1, q_2, p_1, p_2) are not variables of ”action – angle” type, since K_2 depends not only on p_1, p_2, but also on the phase coordinates q_1, q_2. Therefore, one must further pass from the canonical variables (q_1, q_2, p_1, p_2) to the new canonical variables $(\theta_1, \theta_2, \tau_1, \tau_2)$ according to the Birkhoff formulas [5]

$$\begin{aligned} q_1 = \sqrt{2\tau_1}\sin\theta_1, \quad q_2 = \sqrt{2\tau_2}\sin\theta_2, \\ p_1 = \sqrt{2\tau_1}\cos\theta_1, \quad p_2 = \sqrt{2\tau_2}\cos\theta_2, \end{aligned} \tag{14}$$

The transformation (14) ”eliminates” expressions with the new angle coordinates θ_1, θ_2 from the quadratic part of the new Hamiltonian F

$$K(q_1, q_2, p_1, p_2) \to F(\theta_1, \theta_2, \tau_1, \tau_2).$$

In other words, if one represents the new Hamiltonian F in the form

$$F(\theta_1, \theta_2, \tau_1, \tau_2) = F_2(\tau_1, \tau_2) + F_3(\theta_1, \theta_2, \tau_1, \tau_2) + F_4(\theta_1, \theta_2, \tau_1, \tau_2) + ...,$$

then its quadratic form F_2 should not depend on the phase angles θ_1, θ_2, but must depend only on the new momenta τ_1, τ_2. After the substitution (14) in expressions (11)–(13), we will have the following equalities for the forms F_2, F_3, and F_4:

$$F_2 = 0.774284\tau_1 - 0.618792\tau_2,$$

$$\begin{aligned}
F_3 &= (11.425\cos\theta_1 + 46.8457\cos 3\theta_1 + 22.6055\sin\theta_1 + 7.89204\sin 3\theta_1)\tau_1^{3/2} \\
&+ (21.6884\cos(2\theta_1+\theta_2) + 14.1864\cos\theta_2 + 137.377\cos(2\theta_1-\theta_2) \\
&+ 29.9068\sin(2\theta_1+\theta_2) - 44.5182\sin\theta_2 + 46.4992\sin(2\theta_1-\theta_2))\tau_1\tau_2^{1/2} \\
&+ (31.7316\cos\theta_1 + 3.03601\cos(\theta_1+2\theta_2) + 130.34\cos(\theta_1-2\theta_2) \\
&+ 62.7842\sin\theta_1 - 20.2224\sin(\theta_1+2\theta_2) + 68.8283\sin(\theta_1-2\theta_2))\tau_1^{1/2}\tau_2 \\
&+ (9.85026\cos\theta_2 + 39.8211\cos 3\theta_2 - 30.911\sin\theta_2 - 29.9792\sin 3\theta_2)\tau_2^{3/2},
\end{aligned}$$

$$\begin{aligned}
F_4 &= (-166.062 - 193.301\cos 2\theta_1 + 66.3508\cos 4\theta_1 + 262.425\sin 2\theta_1 \\
&+ 213.689\sin 4\theta_1)\tau_1^2 + (-736.077\cos(\theta_1+\theta_2) - 182.809\cos(3\theta_1+\theta_2) \\
&- 738.186\cos(\theta_1-\theta_2) + 130.436\cos(3\theta_1-\theta_2) + 118.39\sin(\theta_1+\theta_2) \\
&+ 355.128\sin(3\theta_1+\theta_2) + 725.101\sin(\theta_1-\theta_2) + 915.061\sin(3\theta_1-\theta_2))\tau_1^{3/2}\tau_2^{1/2} \\
&+ (-831.7 - 748.749\cos 2\theta_1 - 401.159\cos(2\theta_1+2\theta_2) - 895.925\cos 2\theta_2 \\
&- 25.6232\cos(2\theta_1-2\theta_2) + 1016.5\sin 2\theta_1 + 132.471\sin(2\theta_1+2\theta_2) \\
&- 635.537\sin 2\theta_2 + 1432.63\sin(2\theta_1-2\theta_2))\tau_1\tau_2 + (-350.012\cos(\theta_1+3\theta_2) \\
&- 924.689\cos(\theta_1+\theta_2) - 951.62\cos(\theta_1-\theta_2) - 174.31\cos(\theta_1-3\theta_2) \\
&- 172.329\sin(\theta_1+3\theta_2) + 148.726\sin(\theta_1+\theta_2) + 934.752\sin(\theta_1-\theta_2) \\
&+ 972.492\sin(\theta_1-3\theta_2))\tau_1^{1/2}\tau_2^{3/2} - (261.892 + 384.449\cos 2\theta_2 \\
&+ 84.507\cos 4\theta_2 + 272.715\sin 2\theta_2 + 241.328\sin 4\theta_2)\tau_2^2.
\end{aligned}$$

The canonical variables $(\theta_1, \theta_2, \tau_1, \tau_2)$ are variables of "action–angle" type, and the Hamiltonian equations in the neighborhood of the equilibrium point S_1 are expressed by equalities

$$\begin{aligned}
\frac{d\tau_1}{dt} &= -\frac{\partial F_3}{\partial\theta_1} - \frac{\partial F_4}{\partial\theta_1} + ..., \quad \frac{d\theta_1}{dt} = \frac{\partial F_2}{\partial\tau_1} + \frac{\partial F_3}{\partial\tau_1} + \frac{\partial F_4}{\partial\tau_1} + ..., \\
\frac{d\tau_2}{dt} &= -\frac{\partial F_3}{\partial\theta_2} - \frac{\partial F_4}{\partial\theta_2} + ..., \quad \frac{d\theta_2}{dt} = \frac{\partial F_2}{\partial\tau_2} + \frac{\partial F_3}{\partial\tau_2} + \frac{\partial F_4}{\partial\tau_2} + ...,
\end{aligned} \tag{15}$$

Unfortunately the Hamiltonian equations (15) still do not fulfil the conditions of the Arnold – Moser theorem. It is necessary to construct another canonical transformation $(\theta_1, \theta_2, \tau_1, \tau_2) \to (\psi_1, \psi_2, T_1, T_2)$ that will "annihilate" the form of order 3/2, i.e., transform $F_3(\theta_1, \theta_2, \tau_1, \tau_2)$ to $W_3(\psi_1, \psi_2, T_1, T_2) = 0$ and the second–order form $F_4(\theta_1, \theta_2, \tau_1, \tau_2)$ to $W_4(T_1, T_2)$.

We will search for the last canonical transformation (with the required accuracy) in the form

$$\begin{aligned}
\theta_1 &= \psi_1 + V_{13}(\psi_1, \psi_2, T_1, T_2) + V_{14}(\psi_1, \psi_2, T_1, T_2), \\
\theta_2 &= \psi_2 + V_{23}(\psi_1, \psi_2, T_1, T_2) + V_{24}(\psi_1, \psi_2, T_1, T_2), \\
\tau_1 &= T_1 + U_{13}(\psi_1, \psi_2, T_1, T_2) + U_{14}(\psi_1, \psi_2, T_1, T_2),
\end{aligned}$$

$$\tau_2 = T_2 + U_{23}(\psi_1, \psi_2, T_1, T_2) + U_{24}(\psi_1, \psi_2, T_1, T_2),$$

where $U_{13}, U_{23}, U_{14}, U_{24}, V_{13}, V_{23}, V_{14}, V_{24}$ are determined from some linear partial differential equations. For their solution, we apply the method of asymptotic integration of multifrequency systems of differential equations, developed in [18]. For example, the equation for the unknown function U_{13} has the form

$$\frac{\partial U_{13}}{\partial \psi_1}\sigma_1 - \frac{\partial U_{13}}{\partial \psi_2}\sigma_2 = A_{13}(\psi_1, \psi_2, T_1, T_2), \tag{16}$$

where
$A_{13} = (11.425 sin\psi_1 + 140.537 \sin 3\psi_1 - 22.6055 \cos\psi_1 - 23.6761 \cos 3\psi_1)T_1^{3/2}$
$+ (43.3768 \sin(2\psi_1 + \psi_2) + 274.753 \sin(2\psi_1 - \psi_2) - 59.8137 \cos(2\psi_1 + \psi_2)$
$- 92.9983 \cos(2\psi_1 - \psi_2))T_1 T_2^{1/2} + (31.7316 \sin\psi_1 + 3.03601 \sin(\psi_1 + 2\psi_2)$
$+ 130.34 \sin(\psi_1 - 2\psi_2) - 62.7842 \cos\psi_1 + 20.2224 \cos(\psi_1 + 2\psi_2)$
$- 68.8283 \cos(\psi_1 - 2\psi_2))T_1^{1/2} T_2.$

From all solutions of equation (16), it is necessary to choose one which ensures the form (3) of the new Hamiltonian,

$$W_2 = \sigma_1 T_1 - \sigma_2 T_2, \quad W_4 = c_{20} T_1^2 + c_{11} T_1 T_2 + c_{02} T_2^2,$$

where

$$\sigma_1 = 0.774284, \quad \sigma_2 = 0.618792,$$
$$c_{20} = 101.693, \quad c_{11} = 522.084, \quad c_{02} = 168.211.$$

Such solution exists and has the form
$U_{13} = -(14.7556 \cos\psi_1 + 60.5019 \cos 3\psi_1 + 29.1954 \sin\psi_1 + 10.1927 \sin 3\psi_1)T_1^{3/2}$
$- (126.769 \cos(2\psi_1 - \psi_2) + 42.9086 \sin(2\psi_1 - \psi_2) + 46.377 \cos(2\psi_1 + \psi_2)$
$+ 64.3313 \sin(2\psi_1 + \psi_2))T_1 T_2^{1/2} - (40.9819 \cos\psi_1 + 81.0867 \sin\psi_1$
$+ 64.7856 \cos(\psi_1 - 2\psi_2) + 34.2112 \sin(\psi_1 - 2\psi_2) - 6.55302 \cos(\psi_1 + 2\psi_2)$
$+ 43.6486 \sin(\psi_1 + 2\psi_2))T_1^{1/2} T_2.$

Thus, the expansion of the Hamiltonian of the restricted six–body problem in the neighborhood of the equilibrium S_1 with coordinates

$$x^* = y^* = 0.709007$$

presented finally in terms of the canonical variables $(\psi_1, \psi_2, T_1, T_2)$, fulfils all the conditions of the Arnold–Moser theorem, consequently, the equilibrium point S_1 is stable in Lyapunov sense.

In the interval $0 \le m \le 0.0094$, there exist two "resonant" values of the parameter m [15] $(m_1 \approx 0.005, m_2 \approx 0.0035)$ for which the problem of the Lyapunov stability remains open.

Similarly, we have studied all equilibrium bisectorial points of restricted gravitational models, indicated in quote board.

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