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New constructive methods for analysis of resonant systems

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Abstract. The modern theory of perturbations, based on the Krylov–Bogolyubov method [1], has two essential advantages: the determination of the iterations does not require the preliminary solution of the generating equation and the choice of the initial conditions, which for every approximation minimizes the difference "exact solution minus asymptotic solution". The algorithm of constructing the perturbed solution may be realized with computer algebra methods.

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1 The classical perturbation theory

Consider an n–dimensional differential equation with small parameter

$$\frac{dz}{dt} = Z(z,t,\mu), \qquad z(0) = z_0, \tag{1}$$

where μ is the small parameter, while the vector function $Z(z,t,\mu)$ is the known and has properties which ensure the existence and uniqueness of solutions of the Cauchy problem (1) in the $(n+1)$–dimensional domain $G_{n+1} = \{(z,t) \in G_n \times R\}$ of the Euclidean space.

Our purpose is to construct this solution [2]. Along with (1) we consider the equivalent equation

$$\frac{dz}{dt} = \overline{Z}(z,t,\mu) + Z(z,t,\mu) - \overline{Z}(z,t,\mu), \qquad z(0) = z_0, \tag{2}$$

where $\overline{Z}(z,t,\mu)$ is an arbitrary function. We write the linear equality

$$z(t,\mu) = \overline{z}(t,\mu) + u(t,\mu), \tag{3}$$

where $\overline{z}$ and u are some new unknown functions. The solution of Cauchy problem for (1) can be found by solving the following two Cauchy problems:

$$\frac{d\overline{z}}{dt} = \overline{Z}(\overline{z},t,\mu), \qquad \overline{z}(0) = \overline{z}_0 \in G_n, \tag{4}$$

$$\frac{du}{dt} = Z(\overline{z} + u, t, \mu) - \overline{Z}(\overline{z}, t, \mu), \qquad u(0) = z_0 - \overline{z}_0, \tag{5}$$

where $\overline{z}_0$ is some new initial point. Equation (4) defines the choice of the initial approximation $\overline{z}(t, \mu)$ for the exact solution $z(t, \mu)$ of the problem (1), while equation (5) defines the total perturbation $u(t, \mu)$. From the problem (5) one can see that the perturbation $u(t, \mu)$ depends on the choice of the function $\overline{Z}(\overline{z}, t, \mu)$ and on the initial point $\overline{z}_0$, and, moreover, its finding is possible only after the solution of equation (4). Thus, for the Cauchy problem (1) it is possible to construct a set of variants of the perturbation theory with the parameters $\overline{Z}$ and $\overline{z}_0$. It does not mean at all that the function $\overline{Z}(\overline{z}, t, \mu)$ and the initial point z_0 may be chosen arbitrarily. It seems to be reasonable that the function $\overline{Z}(\overline{z}, t, \mu)$ would be chosen to have a possibly simpler analytic structure. On the other hand, the solutions of equation (5) must be "small" under the norm.

In classical works on nonlinear oscillations and cosmic dynamics there were commonly used three schemes:

$$\begin{cases} \frac{d\overline{z}}{dt} = A(t)\overline{z}, \quad \overline{z}(0) = z_0, \\ \frac{du}{dt} = Z(\overline{z} + u, t, \mu) - A(t)\overline{z}, \quad u(0) = 0, \end{cases} \tag{6}$$

$$\begin{cases} \frac{d\overline{z}}{dt} = Z(\overline{z}, t, 0), \quad \overline{z}(0) = z_0, \\ \frac{du}{dt} = Z(\overline{z} + u, t, \mu) - Z(\overline{z}, t, 0), \quad u(0) = 0, \end{cases} \tag{7}$$

$$\begin{cases} \frac{d\overline{z}}{dt} = \overline{Z}(\overline{z}, t, \mu), \quad \overline{z}(0) = z_0, \\ \frac{du}{dt} = Z(\overline{z} + u, t, \mu) - \overline{Z}(\overline{z}, t, \mu), \quad u(0) = 0. \end{cases} \tag{8}$$

Equations (6) represent the linearization method, equations (7) characterize the small parameter method, while equations (8) feature the averaging method, provided that the generator $\overline{Z}$ is constructed on the basis of some averaging operator.

The main idea of the classical perturbation theory (that is, the solution of the problems (4) and (5)) is that the solution of the generating equation (4) is being constructed by means of a finite number of analytic procedures or by numerical methods, after solving of the equation for perturbations (5) by means of any iterative method, symbolically designated by

$$\frac{du_k}{dt} = Z(\overline{z}(t, \mu) + u_{k-1}(t, \mu), t, \mu) - \overline{Z}(\overline{z}(t, \mu), \mu), \tag{9}$$

with $u_k(0) = z_0 - \overline{z}_0$ and $k = 1, 2, \ldots$.

2 New variants of the perturbation theory

Now we assume that the perturbation u depends on $\overline{z}, t$, and μ, that is instead of (3) we have the equality

$$z(t, \mu) = \overline{z}(t, \mu) + u(\overline{z}, t, \mu). \tag{10}$$

This equality represents the transformation from the phase space $\{z\}$ to the new phase $\{\overline{z}\}$ $(\{z\} \to \{\overline{z}\})$ and the inverse transformation $(\{\overline{z}\} \to \{z\})$ if the Jacobian matrix $\partial u/\partial \overline{z}$ is nonsingular.

The following differential equation holds

$$\frac{dz}{dt} = \frac{d\overline{z}}{dt} + \left(\frac{\partial u}{\partial \overline{z}}, \frac{d\overline{z}}{dt}\right) + \frac{\partial u}{\partial t}, \tag{11}$$

where $(\partial u/\partial \overline{z}, d\overline{z}/dt)$ is the product between the matrix $\partial u/\partial \overline{z}$ and the vector $d\overline{z}/dt$. Therefore, instead of equations (4) and (5) of the classical perturbation theory, we shall have the equations [2,3]

$$\frac{d\overline{z}}{dt} = \overline{Z}(\overline{z}, t, \mu), \qquad \overline{z}(0) = \overline{z}_0, \tag{4}$$

$$\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial \overline{z}}, \overline{Z}(z, t, \mu)\right) = Z(\overline{z} + u, t, \mu) - \overline{Z}(\overline{z}, t, \mu), \qquad u(0) = z_0 - \overline{z}_0. \tag{12}$$

The perturbation theory based on equations (4) and (12) differs from the classical perturbation theory in an essential point: the determination of the perturbation $u(\overline{z}, t, \mu)$ from equation (12) – called by us generalized Krylov–Bogolyubov equation [3,4] – does not require the preliminary solving of the generating equation (4). This allows us to determine the perturbation and the initial approximation independently from each other, and therefore the accuracies of their determination are independent, too. This is impossible within the framework of the classical theory of perturbations.

The equation (12) constitutes the Cauchy problem for a quasilinear n–dimensional system of partial differential equations of first order with respect to the n–dimensional perturbation vector u. Its solution can be found by the methods of characteristics or by Cauchy's method. This equation was considered for the first time in a work of Bogolyubov [1] while tackling a question of applicability of the averaging method to a special class of ordinary differential equations.

The asymptotic theory of equation (12) for problems belonging to celestial mechanics was developed in the textbooks of Grebenikov [4] and Grebenikov and Mitropolsky [5]. We mean those problems of dynamics which are described by multifrequential systems of differential equations given on tori, and – in particular – by Hamiltonian systems with variables of the type action–angle and with the Hamiltonian periodic on the angular variable.

So, let a problem of celestial mechanics be described by a multifrequential system of the $(m+n)$–th order

$$\frac{dx}{dt} = \mu X(x, y), \tag{13.1}$$

$$\frac{dy}{dt} = \omega(x) + \mu Y(x, y), \tag{13.2}$$

where x and X are m–dimensional vectors, y, Y, and ω are n–dimensional vectors, $\omega(x)$ is the vector of frequencies, and we assume that $X(x,y)$ and $Y(x,y)$ are 2π–periodic functions with respect to y. Then they are represented by the n–multiple Fourier series

$$X(x,y) = \sum F_k(x,y), \tag{14.1}$$

$$Y(x,y) = \sum G_k(x,y), \tag{14.2}$$

where $F_k(x,y) = X_k(x)e^{i(k,y)}$, $G_k(x,y) = Y_k(x)e^{i(k,y)}$, $i = \sqrt{-1}, (k,y) = \sum\limits_{s=1}^{n} k_s y_s$, $\sum$ abridges $\sum\limits_{\|k\|\in I}$, $\|k\| = \sum\limits_{s=1}^{n} \mid k_s \mid$, $k_s = 0, \pm 1, ...$, and $I = \{0, 1, 2, ...\}$.

Let us apply to system (13) the above stated idea of constructing a modern perturbation theory using asymptotic expansions with respect to the small parameter μ.

We choose for (13) a generating system of the form

$$\frac{d\overline{x}}{dt} = \mu\overline{X}(\overline{x},\overline{y}) + \sum_{k\geq 2}\mu^k A_k(\overline{x},\overline{y}), \tag{15.1}$$

$$\frac{d\overline{y}}{dt} = \omega(x) + \mu\overline{Y}(\overline{x},\overline{y}) + \sum_{k\geq 2}\mu^k B_k(\overline{x},\overline{y}), \tag{15.2}$$

where $\overline{X}, \overline{Y}, A_k, B_k$ are arbitrary functions of their arguments.

Let us look for the replacement of the variables (10) as formal series

$$x = \overline{x} + \sum_{k\geq 1}\mu^k u_k(\overline{x},\overline{y}), \tag{16.1}$$

$$y = \overline{y} + \sum_{k\geq 2}\mu^k v_k(\overline{x},\overline{y}), \tag{16.2}$$

with unknown functions $u_k(\overline{x},\overline{y}), v_k(\overline{x},\overline{y})$. After differentiating (16) and taking into account (13) and (15), to determine the transformation functions u_k and v_k, we have an infinite system of linear partial differential equations of first order

$$\left(\frac{\partial u_1}{\partial \overline{y}}, \omega(\overline{x})\right) = X(\overline{x},\overline{y}) - \overline{X}(\overline{x},\overline{y}), \tag{17.1}$$

$$\left(\frac{\partial v_1}{\partial \overline{y}}, \omega(\overline{x})\right) = \left(\frac{\partial \omega}{\partial \overline{x}}, u_1\right) + Y(\overline{x},\overline{y}) - \overline{Y}(\overline{x},\overline{y}), \tag{17.2}$$

$$\left(\frac{\partial u_k}{\partial \overline{y}}, \omega(\overline{x})\right) = \Phi_k(\overline{x},\overline{y}, u_1, v_1, ..., v_{k-1}, u_{k-1}A_2, B_2, ..., A_k), \tag{17.3}$$

$$\left(\frac{\partial v_k}{\partial \overline{y}}, \omega(\overline{x})\right) = \Psi_k(\overline{x},\overline{y}, u_1, v_1, ..., v_{k-1}, u_k A_2, B_2, ..., A_k, B_k), \tag{17.4}$$

$$k = 2, 3, \ldots \ .$$

The system (17) has a remarkable property: it is possible to integrate it analytically [3, 4] for any vector-index k if for the functions $\overline{X}$ and $\overline{Y}$ we choose some averages of the functions X and Y.

Indeed, let the generators $\overline{X}(\overline{x}, \overline{y}), \overline{Y}(\overline{x}, \overline{y})$ be the partial sums of series (14)

$$\overline{X}(\overline{x}, \overline{y}) = \sum_{1} F_k(\overline{x}, \overline{y}), \tag{18.1}$$

$$\overline{Y}(\overline{x}, \overline{y}) = \sum_{2} G_k(\overline{x}, \overline{y}), \tag{18.2}$$

where $\sum\limits_{j}$ abridges $\sum\limits_{\|k\| \in I_j}$, while I_1 and I_2 are subsets of integer nonnegative numbers from the set of all nonnegative integers I. In particular, I_1 or I_2 may consist of only one number, zero; that means

$$\overline{X}(\overline{x}, \overline{y}) = (2\pi)^{-n} \int\limits_{0}^{2\pi} X(\overline{x}, \overline{y}) d\overline{y}_1, \ldots, d\overline{y}_n. \tag{19}$$

Usually subsets I_1 and I_2 are "resonant": for $\|k\| \in I_j$

$$(k, \omega(x)) = 0.$$

If $\overline{X}$ and $\overline{Y}$ are chosen according to (18), then

$$X(\overline{x}, \overline{y}) - \overline{X}(\overline{x}, \overline{y}) = \sum_{*} F_k(\overline{x}, \overline{y}), \tag{20.1}$$

$$Y(\overline{x}, \overline{y}) - \overline{Y}(\overline{x}, \overline{y}) = \sum_{**} G_k(\overline{x}, \overline{y}), \tag{20.2}$$

where $\sum\limits_{*}$ abridges $\sum\limits_{\|k\| \in I - I_1}$, and $\sum\limits_{**}$ abridges $\sum\limits_{\|k\| \in I - I_2}$.

Using the method of characteristics, it is possible to find the exact solution of (17.1)–(17.2):

$$u_1(\overline{x}, \overline{y}) = \sum_{*} F_k^{*}(\overline{x}, \overline{y}) + \varphi_1(\overline{x}), \tag{21.1}$$

$$v_1(\overline{x}, \overline{y}) = \sum_{**} G_k^{*}(\overline{x}, \overline{y}) + \left(\frac{\partial \omega(\overline{x})}{\partial \overline{x}}, \sum_{*} F_k^{**}(\overline{x}, \overline{y}) \right) + \left(\left(\frac{\partial u_1}{\partial \overline{x}}, \varphi_1(\overline{x}) \right), \overline{y} \right) + \psi_1(\overline{x}), \tag{21.2}$$

where

$$F_k^*(\overline{x},\overline{y}) = \frac{F_k(\overline{x},\overline{y})}{f_k(\overline{x})}, \quad G_k^*(\overline{x},\overline{y}) = \frac{G_k(\overline{x},\overline{y})}{f_k(\overline{x})},$$

$$F_k^{**}(\overline{x},\overline{y}) = \frac{F_k(\overline{x},\overline{y})}{(f_k(\overline{x}))^2}, \quad f_k(\overline{x}) = i(k,\omega(\overline{x})),$$

while φ_1, ψ_1 are arbitrary differentiable functions of their arguments $\overline{x}_1, ..., \overline{x}_m$.

The integration of equations (17) for $k = 2, 3, ...$ is not very difficult, therefore the functions $u_2, v_2, ...$ are also presented by means of known analytic expressions [3, 4]. Rather important is the fact that while determining the functions u_2 and v_2 (those are perturbations of second order) we can use the functions $A_2, B_2, \varphi_1, \psi_1$.

By (21) one can see that if $\varphi_1 \neq 0$ then v_1 will be growing similary to the linear function t, because $\overline{y} \sim t$. Hence for the perturbations $u_1, v_1, u_2, v_2, ...$ to have an "oscillatory" but not a "rapidly growing" character it is necessary that

$$\varphi_k(\overline{x}) \equiv 0, \psi_k(\overline{x}) \equiv 0, \qquad k = 1, 2, ... \tag{22}$$

In their turn, these equalities show that the "best" perturbation theory is obtained when the generating equations and the perturbation equations are solved for other initial conditions in comparison with the initial equations. Indeed, if $\varphi_1(\overline{x}_0) = 0, \psi_1(\overline{x}_0) = 0$, it is easy to see, by (21), that $u_1(\overline{x}_0, \overline{y}_0) \neq 0, v_1(\overline{x}_0, \overline{y}_0) \neq 0$ and, with an accurate μ, the new initial conditions $(\overline{x}_0, \overline{y}_0)$ are connected with (x_0, y_0) by means of the functional equations

$$x_0 = \overline{x}_0 + \mu u_1(\overline{x}_0, \overline{y}_0), \tag{23.1}$$

$$y_0 = \overline{y}_0 + \mu v_1(\overline{x}_0, \overline{y}_0). \tag{23.2}$$

Similar equations for new initial conditions $(\overline{x}_0, \overline{y}_0)$ can be derived for the perturbation theory of any order k:

$$x_0 = \overline{x}_0 + \sum_{s=1}^{k} \mu^s u_s(\overline{x}_0, \overline{y}_0), \tag{24.1}$$

$$y_0 = \overline{y}_0 + \sum_{s=1}^{k} \mu^s v_s(\overline{x}_0, \overline{y}_0). \tag{24.2}$$

This is a second essential difference of the modern perturbation theory from the classical one, in which it is difficult to dispose by choice of the initial point z_0.

If we construct the perturbation theory of second order, that is, we write a system (17) for $k = 2$, we shall have

$$\left(\frac{\partial u_2}{\partial \overline{y}}, \omega(\overline{x}) \right) = \Phi_2(\overline{x}, \overline{y}, u_1, v_1, A_2), \tag{25.1}$$

$$\left(\frac{\partial v_2}{\partial \overline{y}}, \omega(\overline{x})\right) = \Psi_2(\overline{x}, \overline{y}, u_1, v_1, u_2, A_2, B_2). \tag{25.2}$$

These equations include the arbitrary functions A_2, B_2, and the best way is to choose them such that

$$\int_0^{2\pi} \ldots \int_0^{2\pi} \Phi_2 d\overline{y}_1, \ldots, d\overline{y}_n = 0, \tag{26.1}$$

$$\int_0^{2\pi} \ldots \int_0^{2\pi} \Psi_2 d\overline{y}_1, \ldots, d\overline{y}_n = 0. \tag{26.2}$$

These conditions guarantee us a choice of solutions u_2 and v_2, which would also be of oscillatory character. This statement holds provided that the functions φ_2 and ψ_2 (by analogy with φ_1 and ψ_1) are chosen identically equal to zero.

The stated analytic algorithm means that we construct successively the replacement of variables

$$(x, y) \to (\overline{x}_1, \overline{y}_1) \to (\overline{x}_2, \overline{y}_2) \to \ldots \to (\overline{x}_s, \overline{y}_s),$$

where

$$x_s = \overline{x} + \sum_{k=1}^{s} \mu^k u_k(\overline{x}, \overline{y}), \tag{27.1}$$

$$y_s = \overline{y} + \sum_{k=1}^{s} \mu^k v_k(\overline{x}, \overline{y}), \tag{27.2}$$

From the geometric point of view, the chain written above means the successive transformation of the initial phase space $\{x, y\}$ into the new phase space, in which the problem of perturbation determination of any order becomes analytically solvable.

Naturally, for the final construction of the solution of the initial equations (13), one has to solve the generating equation of the corresponding order s

$$\frac{d\overline{x}_s}{dt} = \mu \overline{X}(\overline{x}_s, \overline{y}_s) + \sum_{k=2}^{s} \mu^k A_k(\overline{x}_s), \tag{28.1}$$

$$\frac{d\overline{y}_s}{dt} = \omega(\overline{x}_s) + \mu \overline{Y}(\overline{x}_s, \overline{y}_s) + \sum_{k=2}^{s} \mu^k B_k(\overline{x}_s), \tag{28.2}$$

with the initial conditions $\overline{x}_s(0), \overline{y}_s(0)$ from equalities (24) and then, by means of (16), one can find an approximation s to

$$x_s(t, \mu) = \overline{x}_s(t, \mu) + \sum_{k=1}^{s} \mu^k u_k(\overline{x}_s, \overline{y}_s), \tag{29.1}$$

$$y_s(t,\mu) = \overline{y}_s(t,\mu) + \sum_{k=1}^{s} \mu^k v_k(\overline{x}_s, \overline{y}_s). \tag{29.2}$$

In conclusion, we want to note once again that in formulae (29) the functions u_k, v_k are found by analytic methods and, if the solution of the generating equation (28) can also be found through analytic methods, this is the best we can have in the nonlinear analysis. If this is not possible, then the combination of numerical methods with analytic ones applied to perturbation equations gives sometimes a large gain of economies of computer resources.

Finally, we will discuss the problems which can be solved by the contemporary methods of computer algebra.

1) The constructing of the averaging functions $\overline{X}(x,y), \overline{Y}(x,y)$.

First we calculate the initial frequencies $\omega_1(x_0), \omega_2(x_0), \ldots, \omega_n(x_0)$ and then we calculate the subsets of the integer numbers $I_1 \times I_2$, marking the proper k inequality vector

$$|(k, \omega(x_0))| < \varepsilon_1, \qquad |(k, \omega(x_0))| < \varepsilon_2.$$

If $\varepsilon_1 = \varepsilon_2$, $I_1 = I_2$. The ε_1 and ε_2 values are given apriori.

2) Afterwards, we calculate the perturbations of the first order $u_1(\overline{x}, \overline{y})$ and $v_1(\overline{x}, \overline{y})$ from equalities (21.1), (21.2).

3) The most arduous work is done while constructing Ψ_2 and Φ_2 functions, thanks to which we can calculate the functions of the second approximation u_2 and v_2 from equations (25.1), (25.2). It consists in multiplying Fourier series and assigning the resonant parts from the resulting products. Those resonant parts define the unknown functions A_2 and B_2.

4) If scientific researcher limits himself to the asymptotic theory of the second order, which is solving system (1) in the form

$$\begin{aligned} x(t,\mu) &= \overline{x}(t,\mu) + \mu u_1(\overline{x}(t,\mu),\, \overline{y}(t,\mu)) + \mu^2 u_2(\overline{x}(t,\mu),\, \overline{y}(t,\mu)), \\ y(t,\mu) &= \overline{y}(t,\mu) + \mu v_1(\overline{x}(t,\mu),\, \overline{y}(t,\mu)) + \mu^2 v_2(\overline{x}(t,\mu),\, \overline{y}(t,\mu)), \end{aligned} \tag{30}$$

the initial conditions $\overline{x}(0,\mu)$ and $\overline{y}(0,\mu)$ for the solution of the generator system

$$\frac{d\overline{x}}{dt} = \mu \overline{X}(\overline{x}, \overline{y}) + \mu^2 A_2(\overline{x}), \quad \frac{d\overline{y}}{dt} = \omega(\overline{x}) + \mu \overline{Y}(\overline{x}, \overline{y}) + \mu^2 B_2(\overline{x}), \tag{31}$$

have to be calculated from nonlinear functional equations

$$\begin{aligned} \overline{x}(0,\mu) &= x(0,\mu) - \mu u_1(\overline{x}(0,\mu),\, \overline{y}(0,\mu)) - \mu^2 u_2(\overline{x}(0,\mu),\, \overline{y}(0,\mu)), \\ \overline{y}(0,\mu) &= y(0,\mu) - \mu v_1(\overline{x}(0,\mu),\, \overline{y}(0,\mu)) - \mu^2 v_2(\overline{x}(0,\mu),\, \overline{y}(0,\mu)). \end{aligned} \tag{32}$$

The solutions of the system of equations (32) are to be found by means of iterative methods.

Finally, we will emphasize two extraordinary moments of the asymptotic theory based on averaging methods.

1. According to the super N.N. Bogolyubov's idea, the transformed equation (17.3), (17.4) is not given apriori at the beginning, but is constructed at every step of calculations. This is meant to minimalize the deviation of the asymptotic solution from the exact solution of the system (1). Such approach is not present in the classical perturbation theory.

2. The choice of the optimum initial conditions at every step of the constructing process improves the theory and the application practice of the resonant systems of differential equations.

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