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Quadratic systems with limit cycles of normal size

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Abstract. In the class of planar autonomous quadratic polynomial differential systems we provide 6 different phase portraits having exactly 3 limit cycles surrounding a focus, 5 of them have a unique focus. We also provide 2 different phase portraits having exactly 3 limit cycles surrounding one focus and 1 limit cycle surrounding another focus. The existence of the exact given number of limit cycles is proved using the Dulac function. All limit cycles of the given systems can be detected through numerical methods; i.e. the limit cycles have "a normal size" using Perko's terminology.

Mathematics subject classification: 34C07, 34C08.
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1 Introduction

A planar autonomous quadratic polynomial differential system (or simply a *quadratic system*) in what follows is a system of the form

$$\frac{dx}{dt} = \sum_{i+j=0}^{2} a_{ij}x^i y^j \equiv P(x,y), \quad \frac{dy}{dt} = \sum_{i+j=0}^{2} b_{ij}x^i y^j \equiv Q(x,y), \tag{1}$$

with $a_{ij}, b_{ij} \in \mathbb{R}$. It is known (see, for instance [17]) that a quadratic system can have only limit cycles enclosing a unique singular point, which is a focus. As system (1) has no more than two foci [17], only the following distributions of limit cycles are allowed: n, (n_1, n_2), where $n \in \mathbb{N}$, and $n_1, n_2 \in \mathbb{N} \cup \{0\}$ with $n_1 + n_2 > 0$. Here n is the number of limit cycles surrounding a focus provided that system (1) has only one focus, and n_1 and n_2 are the number of limit cycles surrounding every one of the two foci provided that the system has exactly two foci. Recently, Zhang Pingguang [20, 21] has proved that if $n_i > 0$ for $i = 1, 2$, then either $n_1 = 1$, or $n_2 = 1$.

The following distributions of limit cycles for quadratic systems (1) are known:

(a) 1 and $(1,0)$;	(b) 2 and $(2,0)$;	(c) 3 and $(3,0)$;
(d) $(1,1)$;	(e) $(2,1)$;	(f) $(3,1)$.

With the help of distinct results on uniqueness of a limit cycle (see [15, 17, 22]), being the most effective result from Zhang Zhifen (see [14]), it has been proved for

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the quadratic systems (1) that the distributions of limit cycles (a) and (d) exist, see [5, 6, 14, 16, 18, 19]. The small class of systems with distribution (b) is obtained by bifurcation of limit cycles either from a focus and from a separatrix cycle, or from a focus, see [17].

The most complicated distributions of limit cycles are the distributions (c), (e) and (f). They are obtained also with the help of bifurcations, and by perturbing quadratic systems (1) having a center [2]. However using these methods it is only possible to obtain infinitesimal limit cycles which, in general, are very difficult to detect using numerical methods. Thus, Perko in the work [14] can exhibit quadratic systems with limit cycles "of normal size" using his terminology, i.e. limit cycles which can be detected easily by numerical methods. The main method used by him, consists in considering a set of systems with a rotating parameter, and in studying the bifurcations of limit cycles under the variation of this parameter. For more details on rotating families see [13, 14, 17, 22], and Section 2.

Perko in [14] provided examples of quadratic systems with the six distributions of limit cycles (a)–(f), but he did not consider all the possible phase portraits with these distributions of limit cycles. The purpose of this paper is: first, to systematize Perko's method; and second, to study different phase portraits with the distributions (c) and (f) of limit cycles.

For proving the existence of the exact given number of limit cycles we shall use Dulac functions, see [17] for more details on these functions. A key point for studying the distributions (c) and (f) of limit cycles are the works [1] and [12], where the qualitative phase portraits of all quadratic systems having a weak focus of third order are classified, and additionally, it is described the partition of the parameter space into domains associated to the different topological phase portraits.

By means of an affine transformation of the phase variables and a change of the time scale, a quadratic system (1) generically can be written as

$$\begin{aligned} \frac{dx}{dt} &= 1 + xy, \\ \frac{dy}{dt} &= a_{00} + a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + ay^2, \end{aligned} \tag{2}$$

where $a_{00} = a_{01} + a_{11} - a_{10} - a_{20} - a$.

In Table 1 we summarize the main results of this paper, i.e. the different configurations of singular points compatible with the distributions (c) and (f) of limit cycles. The results of that table are for quadratic systems in the normal form (2). A focus, a node or a saddle is denoted by F, N and S, respectively. If they are at infinity in the Poincaré Compactification, then they have the subindex ∞. For more details on the Poincaré compactification of a planar polynomial differential system see [8].

N^0	Coefficients of system					Singular points	Cycle distr.
	a	a_{20}	a_{11}	a_{01}	a_{10}		
1	3	-12	-1.398	8.4	15.28	$1F + 1N + 2S_\infty + 1N_\infty$	3
2	1.5	-15	0.79993	3.2	9.17	$2F + 2S_\infty + 1N_\infty$	$(3, 0)$
3	-2	12	10.999	-14	-26.1	$1F + 3S + 3N_\infty$	3
4	-2	-1	9.49965	-12.5	6.955	$1F + 1S + 2N_\infty + 1S_\infty$	3
5	-4	-1	13.9987	-21	12.4	$1F{+}1N{+}2S{+}2N_\infty{+}1S_\infty$	3
6	5	-50	-5.49995	16.5	76.45	$1F{+}2N{+}1S{+}1N_\infty{+}2S_\infty$	3
7	8/11	-12	2.1502	67/220	-26.5	$2F + 1S_\infty$	(3,1)
8	1.04	-120	1.51997	1.56	-79.6	$2F + 2S_\infty + 1A_\infty$	(3,1)

Table 1. Different configurations of singular points compatible with the distributions 3 and $(3, 0)$ for the limit cycles of the quadratic systems.

The paper is organized as follows. The results of Table 1 are proved in Section 3, but previously in Section 2, we present the main definitions and basic results which we shall use in the proofs of the results of Table 1.

2 Main definitions and preliminary results for Lienard systems

The *surface of limit cycles* for the system

$$\frac{dx}{dt} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}, \quad f = (f_1, f_2)^T, \tag{3}$$

is the subset $SLC = \{(x, \alpha) \in \mathbb{R}^2 \times \mathbb{R} : x \in L(\alpha), \quad \alpha \in \mathbb{R}\}$, where $L(\alpha)$ is the subset of the phase plane $\mathbb{R}^2$ formed by limit cycles of system (3) with parameter α.

We remark that if all the limit cycles of system (3) surrounding the singular point $x = 0$ (i.e. $f(0, \alpha) = 0$), intersect the half–axis $x_2 = 0$, $x_1 > 0$ only in one point, then instead of working with the surface of limit cycles it is more convenient to consider the *curve of limit cycles*, denoted by CLC, and formed by the points (x_1, α), where x_1 is the abscissa of the point x belonging to a limit cycle and to the half–axis $x_2 = 0$, $x_1 > 0$ for system (3) with parameter α.

We say that the parameter α *rotates* the vector field $f(x, \alpha)$ associated to system (3), or that α is a *rotating parameter*, if one of the two inequalities

$$(f_1)'_\alpha f_2 - f_1 (f_2)'_\alpha \geq 0 \quad (\leq 0), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R},$$

holds, and the inequality never becomes an identity equal to zero on any limit cycle of $L(\alpha)$. Here, $(f_i)'_\alpha$ denotes the derivative of $f_i(x, \alpha)$ with respect to α for $i = 1, 2$. We remark that for the quadratic systems (2), a_{11} is a rotating parameter.

The condition that this inequality never becomes an identity equal to zero on any limit cycle of $L(\alpha)$ can be easily checked, and means that the limit cycles of system (3) really change their position under the variation of the parameter α. Moreover, if α is a rotating parameter, then $L(\alpha_1) \cap L(\alpha_2) = \emptyset$ if $\alpha_1 \neq \alpha_2$. For more details, see [22].

We assume that we have a system (3) and that α is a rotating parameter. Then, the surface of limit cycles SLC is an open subset of $\mathbb{R}^2 \times \mathbb{R}$. By definition the *Andronov–Hopf function* $F : \cup_{\alpha\in\mathbb{R}} L(\alpha) \to \mathbb{R}$ associates to the points of $L(\alpha)$ the value α. Therefore, the surface of limit cycles is determined by the equation $\alpha = F(x)$ running α in $\mathbb{R}$.

If the limit cycles surrounding the singular point $x = 0$ (i.e. $f(0, \alpha) = 0$), intersect the half–axis $x_2 = 0$, $x_1 > 0$ only in one point, instead of function $F(x)$ it is more convenient to consider the function $\alpha = \varphi(x_1) = F(x)|_{x_2=0,\, x_1>0}$, which provides a full information about the limit cycles of system (3) surrounding the point $x = 0$, and their bifurcations when the parameter α varies. Note that the function $\alpha = \varphi(x_1)$, running α in $\mathbb{R}$, defines a curve of limit cycles for system (3) surrounding the point $x = 0$.

For computing the number of limit cycles of quadratic systems (1) we shall use the following two theorems, see [4, 9]. See also [7].

Theorem 1. *Assume that system (1) is structurally stable in a connected region $\Omega \subset \mathbb{R}^2$. Then, there exist a function $\Psi(x, y) \in C^1(\Omega)$ and a constant $k < 0$, such that the inequality*

$$\Phi = k\,\Psi\, div\, f + \frac{\partial\Psi}{\partial x}P + \frac{\partial\Psi}{\partial y}Q > 0, \qquad f = (P, Q), \tag{4}$$

is satisfied in the region Ω. Moreover, the limit cycles of system (1) do not intersect the set $W = \{(x, y) \in \Omega : \Psi(x, y) = 0\}$, and in every two–dimensional connected subregion of Ω where either $\Psi(x, y) > 0$ or $\Psi(x, y) < 0$, system (1) has at most one limit cycle γ, and if exists, is hyperbolic and stable (respectively unstable) if $k\Psi|_\gamma < 0$ (respectively > 0).

If the function $\Psi(x, y)$ satisfies the condition (4), the function $B(x, y) = |\Psi(x, y)|^{1/k}$ is a Dulac function in each subregion $\Psi(x, y) > 0$ or $\Psi(x, y) < 0$, and we have that $\mathrm{div}(Bf) = \Phi|\Psi|^{1/k-1}(\mathrm{sign}\,\Psi)/k$.

Theorem 2. *Let Ω be a simple connected region where system (1) is defined and has a unique singular point, the antisaddle A with $divf(A) \neq 0$. Assume that there exist a function Ψ and a number $k < 0$ satisfying the assumptions of Theorem 1. Suppose that the equation $\Psi(x, y) = 0$ determines in the region Ω a nest of m of ovals surrounding the point A. Then, in each of the $m - 1$ annulus limited by two adjacent ovals, system (1) has exactly one limit cycle. Moreover, system (1) has in the region Ω at most m limit cycles.*

By Theorem 2 it follows that the ovals are transversal to the vector field associated to system (1), and that the annulus limited by two adjacent ovals satisfies the Bendixson principle, see [17] for more details on the Bendixson principle. An additional m-th limit cycle can exist between the most external oval and the boundary of the region Ω.

For the Lienard system

$$\frac{dx}{dt} = y - F(x), \quad \frac{dy}{dt} = -g(x), \tag{5}$$

the determination of the function $\Psi(x, y)$, satisfying the assumptions of Theorem 2, is easy. Thus, if we look for it in the form

$$\Psi = \sum_{i=1}^{n} \Psi_i(x) y^{n-i}, \tag{6}$$

then the appropriate function Φ of Theorem 1 depends only on x if and only if $\partial\Phi/\partial y \equiv 0$. Then, $\partial\Phi/\partial y \equiv 0$ implies

$$\begin{array}{rlll} \Psi_1 = C_1, & \Psi_2' = kfC_1, & \Psi_2 = kFC_1 + C_2, & F'(x) = f(x) \\ \Psi_i' = kf\Psi_{i-1} + (n-i+2)g\Psi_{i-2} + F\Psi_{i-1}', & \Psi_i = \int \Psi_i'(t)dt + C_i, & i = 3, \dots, n. & \end{array} \tag{7}$$

where the C_i for $i = 1, \dots, n$ are arbitrary constants of integration. Therefore, the function Φ has the form

$$\Phi = -kf\Psi_n - g\Psi_{n-1} - F\Psi_n'. \tag{8}$$

In general, the function Φ is a linear combination

$$\Phi = \sum_{j=1}^{n} C_j \Phi_j(x), \tag{9}$$

of convenient functions $\Phi_i(x)$, obtained from (7) and (8).

Theorem 3. *Suppose that the function $g(x)$ of the Lienard system (5) satisfies that $g(0) = 0$, and that its two nearest zeros at 0 are x_1 and x_2 with $x_1 < 0 < x_2$. Assume that there exist the constants $k < 0$ and C_i for $i = 1, \dots, n$, such that the function Φ given in (9) is positive for $x \in (x_1, x_2)$. Then, system (5) has at most $n/2$ limit cycles surrounding the singular point $(0, 0)$.*

For the existence of the positive function Φ, given by (9), on an interval $[\alpha, \beta]$ with $x_1 < \alpha < 0 < \beta < x_2$, it is sufficient that the inequality

$$\max_{|C| \le 1} \min_{x \in [\alpha, \beta]} \Phi(x, C) = \frac{1}{R} > 0,$$

holds. This is equivalent to the existence of a solution for the following problem of optimization:

$$\Phi(x, C) \le 1, \quad |C| \le R, \quad |C| = \max|C_i|, \quad x_1 \le x \le x_2, \quad 0 < R \to \min. \tag{10}$$

We can obtain an approximate solution of problem (10) on the net points x_i solving the discretized problem

$$\Phi(x_i, C) \le 1, \quad |C| \le R, \quad x_i \in [x_1, x_2] \quad i = 1, \dots, N, \quad 0 < R \to \min. \tag{11}$$

If the number N of net points is sufficiently large and problem (11) has a solution, we can expect that problem (10) has also a solution. Note that numerically it is easy to find a minimum of the function $\Phi(x, C^*)$ on $[x_1, x_2]$, where $C = C^*$ is a solution of the problem (11).

In the study of the limit cycles of system (5), the idea of using a function $V(x, y)$ such that its derivative dV/dt on the solutions of system (5) depends only on x, has been used successfully in [10] and also it was mentioned in [4].

3 Perturbed quadratic systems with a weak focus of order three

Since the straight line $x = 0$ is transversal for the vector field associated to system (2), its limit cycles do not intersect $x = 0$. Therefore, in the half–planes $x < 0$ and $x > 0$ its limit cycles can be studied separately. In the half–plane $x > 0$ the transformation $x = 1/\xi, \quad y = (\tilde{y} - F(\xi))\xi^{-a} - \xi$ writes system (2) into the Lienard system

$$\frac{d\xi}{dt} = \tilde{y} - F(\xi), \qquad \frac{d\tilde{y}}{dt} = -g(\xi), \tag{12}$$

where

$$\begin{aligned} f(\xi) &= (a_{11} + a_{01}\xi - (2a+1)\xi^2)\xi^{a-2}, \\ g(\xi) &= (a_{00} + a_{10}\xi + (a_{00} - a_{11})\xi^2 - a_{01}\xi^3 + a\xi^4)qzx\xi^{2a-3}, \\ F(\xi) &= \int_1^{\xi} f(t)dt = \tilde{P}_2(\xi)\xi^{a-1} - \tilde{P}_2(1). \end{aligned}$$

System (2) has a weak focus or a center at the point $A = (1, -1)$, if the conditions

$$L = 2a - a_{01} - a_{10} - 2a_{20} > 0, \qquad V_1 = a_{11} + a_{01} - 2a - 1 = 0, \tag{13}$$

hold. The last condition says that the divergence of system (2) at A is zero.

Clearly, for $a = 2, 3, \ldots$ system (12) is a Lienard polynomial differential system. Moreover, for $a = -2, -3, \ldots$ system (12), under the transformation $\xi = 1/x$, $\tilde{y} = -y$, goes over to

$$\frac{dx}{dt} = y + \hat{P}_2(x)x^{-a-1} - \hat{P}_2(1), \qquad \frac{dy}{dt} = \hat{P}_4(x)x^{-2a-3}, \tag{14}$$

where $\hat{P}_2(x)$, $\hat{P}_4(x)$ are polynomials. Thus, also system (2) for $a = -2, -3, \ldots$ is reduced to a Lienard polynomial differential system.

Under conditions (13) the multiplicity of the weak focus A of system (2) can be determined by its focal values (also called Lyapunov constants), see for instance [11]. For a integer and $|a| > 1$ these focal values can be calculated using the Lienard polynomial systems (12) or (14), or using [11] for an arbitrary value of a. Thus, for system (2) these focal values are

$$\begin{aligned} V_3 &= W_0 - a_{10}W, \\ V_5 &= (4 - 2a - a_{11})V/W, \\ V_7 &= -(a_{11} + 2a + 1)UV/W, \end{aligned} \tag{15}$$

where

$$\begin{aligned}
W_0 &= a_{11}^2(a+1) + a_{11}(2a^2+a-1) - a_{20}(a_{11}(2a-1) + (2a+1)(2a-3)),\\
W &= -1 + 2a^2 + a(a_{11}-1),\\
V &= -a_{11}^2 a(a+1) + a_{20}(a-1)(2a+1)^2,\\
U &= (8-2a^2)(a_{11}+2a+1)^2 - 35(2a+1)(a_{11}+2a+1) + 35(2a+1)^2.
\end{aligned}$$

In short, under conditions (13) system (2) has at A

(i) a focus of first order if $V_3 \neq 0$, it is stable if $V_3 < 0$, otherwise it is unstable;

(ii) a focus of second order if $V_3 = 0$, $V_5 \neq 0$, it is stable if $V_5 < 0$, otherwise it is unstable;

(iii) a focus of third order if $V_3 = V_5 = 0$, and $V_7 \neq 0$, it is stable if $V_7 < 0$, otherwise it is unstable;

(iv) a center if and only if $V_3 = V_5 = V_7 = 0$.

It is well known that perturbing a weak focus of order i inside the class of quadratic systems, we can obtain i infinitesimal limit cycles surrounding the perturbed focus. Therefore, to look for quadratic systems having three limit cycles surrounding a focus, it is natural to perturb systems (2) having a weak focus of order three.

We assume that $W \neq 0$. Then, the value a_{10} can be determined from $V_3 = 0$, that is $a_{10} = W_0/W$. In particular, we obtain that system (2) has a weak focus of order three at A if

$$\begin{aligned}
&a_{11} = \tilde{a}_{11}^* = 4-2a, \quad a \neq 2, \quad a_{01} = \tilde{a}_{01}^* = 2a+1-\tilde{a}_{11}^*,\\
&a_{10} = \tilde{a}_{10}^* = (6(a^2-a-2) + a_{20}(6a-7))/(1-3a),\\
&(a-3-a_{20})/(1-3a) < 0.
\end{aligned} \tag{16}$$

In short, we note that we have a 2–parameter family of quadratic systems (2) with a weak focus of order three at A, the two parameters are a and a_{20}.

We fix the parameters a and a_{20} of a system (2) having a weak focus of third order at A, and change the parameters a_{11}, a_{01} and a_{10} in order to obtain a quadratic system with one small limit cycle surrounding A, being A a weak focus of second order. We must change the parameters a_{11}, a_{01} and a_{10} in such a way that $V_1 = 0$ and $V_5V_7 < 0$. We note that V_5 must be different from zero in order to have at A a weak focus of second order, and that the signs of V_5 and V_7 must be different, because the stability of A and of the limit cycle must be opposite. Thus, we can obtain a limit cycle passing through a point $(x, -1)$ with $x > 1$ but near 1 choosing adequately the functions $a_{11} = \tilde{a}_{11}(x)$, $a_{01} = \tilde{a}_{01}(x) = 2a+1-\tilde{a}_{11}(x)$ and $a_{10} = \tilde{a}_{10}(x) = W_0/W$. Of course, we have that $\tilde{a}_{11}(1) = \tilde{a}_{11}^*$, $\tilde{a}_{10}(1) = \tilde{a}_{10}^*$. The condition for the birth of such a limit cycle, if $a_{11} = \tilde{a}_{11}^* + \Delta a_{11}$, $a_{01} = \tilde{a}_{01}^* + \Delta a_{01}$ and $a_{10} = \tilde{a}_{10}^* + \Delta a_{10}$ must satisfy the inequality

$$(3a-1)(a-2)\Delta a_{11} > 0. \tag{17}$$

in order to have $V_5V_7 < 0$. Therefore, $\Delta a_{11} > 0$ if $a < 1/3$ or $a > 2$, and $\Delta a_{11} < 0$ if $1/3 < a < 2$.

Suppose that for $x = x_0 > 1$ and for the values $\tilde{a}_{11}(x_0)$, $\tilde{a}_{01}(x_0)$, $\tilde{a}_{10}(x_0)$ we have one limit cycle surrounding A and passing through $(x_0, -1)$ and that A is a weak focus of second order. Now we fix a_{11} and a_{01}, and change the parameter a_{10} in order to obtain a quadratic system having two limit cycles surrounding the focus A, being A a weak focus of first order. Such a system must satisfy $V_3 \neq 0$, $V_3V_5 < 0$ and $V_5V_7 < 0$. We denote by $a_{10} = a_{10}(x)$ with $a_{10}(1) = a_{10}(x_0)$, $a_{11} = \tilde{a}_{11}(x_0)$ and $a_{01} = \tilde{a}_{01}(x_0)$ the parameters of a quadratic system having two limit cycles around A such that the new second limit cycle passes through the point $(x, -1)$. The appropriate Andronov–Hopf function a_{10} will have one extremum. Now, we denote by a_{10}^* the value of a_{10} for which system (2) has two limit cycles surrounding A and being A a weak focus of first order.

Finally, we change the parameter a_{11} starting with value $\tilde{a}_{11}(x_0)$ and remaining fixed the other parameters, so that from the weak focus of first order A bifurcates a third limit cycle. Such perturbed quadratic system must satisfy $V_1 \neq 0$, $V_1V_3 < 0$, $V_3V_5 < 0$ and $V_5V_7 < 0$. Then, by changing a_{11} on some interval system (2) will have three limit cycle, and the appropriate Andronov–Hopf function $a_{11} = AH(x)$ will have two extrema.

We have described the general scheme for obtaining quadratic system (2) with three limit cycles around the focus A, and moving conveniently the parameters a_{11}, a_{01} and a_{10}. The limit cycles (which originally bifurcated from A) are not necessarily small.

In what follows, we shall consider quadratic systems (2) with different configuration of singular points and we shall look for distributions 3 and $(3, 1)$ of limit cycles. The functions $\tilde{a}_{11}(x)$, $a_{10}(x)$ and $AH(x)$ will be found with the help of numerical computations.

Example 1: *A quadratic system with* 1 *focus and* 1 *node, and* 3 *limit cycles surrounding the focus, having at infinity* 2 *saddles and* 1 *antisaddle.* We take $a = 3$, $a_{20} = -12$, $a_{11} = -1.398$, $a_{10} = 15.28$ and $a_{01} = 8.4$. Then system (2) has the focus A. Numerical computations show that the system has at least three limit cycles which pass through points $(x_i, -1)$ with $x_1 = 1.26$, $x_2 = 1.98$ and $x_3 = 3.95$. With the help of Bendixson annuli it is possible to prove these numerical results analytically, but here we shall not do it. It is much more interesting to provide the upper bound on the number of limit cycles. We shall show that this upper bound is 3. For that we shall work with the Lienard polynomial system (12). Doing the translation $\xi = x + 1$, we get again another Lienard polynomial system. For this system we shall search a function $\Psi(x, C)$ of the form (6) with $n = 10$ and $k = -1$ satisfying conditions (7). For the corresponding function $\tilde{\Phi}(x, C) = \Phi(x, C)/(1 + 4G^2)$ with $G = \int_0^x g(t)dt$, where $\Phi(x, C)$ is a function satisfying (8) and (9), we shall solve the problem of optimization (11) on a uniform net in the interval $[-0.8, 0.5]$ with $N = 320$ points. This problem has the solution C_i^* equal to -0.0594107, -0.343784,

-0.828227, -0.879519, 0.301152, 1, 0.0814624, -0.275238, -0.00639951, 0.00721968 for $i = 1, \dots, 10$. All the real roots of the polynomial $\Phi(x, C^*)$ lie in interval $x \leq -1$. Therefore, this function is positive in $(-1, +\infty)$. The equation $\Psi(x, y, C^*) = 0$ determines in the half–plane $x > -1$ three annuli surrounding the focus $O = (0, 0)$ of the last Lienard polynomial system and Theorem 3 can be applied. Then, the considered quadratic system (2) have no more than three limit cycles enclosing the focus A and at least two limit cycles. Taking into consideration the numerical computations it is possible to check that the system has exactly 3 limit cycles around the focus A.

Example 2: *A quadratic system with* 2 *foci, and* 3 *limit cycles surrounding one focus and* 0 *limit cycles around the other focus, having at infinity* 2 *saddles and* 1 *antisaddle.* That is, this system has a distribution $(3, 0)$ for its limit cycles. We take $a = 1.5$, $a_{20} = -15$, $a_{11} = 0.79993$, $a_{10} = 9.17$ and $a_{01} = 3.2$. The corresponding system (2) has the foci A and $B = (x_0, -1/x_0)$ with $x_0 = -0.73$. In addition, there are at least 3 limit cycles around the focus A which pass through the points $(x_i, -1)$ for $x_1 = 1.4$, $x_2 = 1.9$ and $x_3 = 3.1$. Now, we show that this system has no more 3 limit cycles around the focus A. We consider a Lienard system (12) and a function $\Psi(\xi, \tilde{y}, C)$ as in (6) and (7) with $n = 11$, $k = -1$ and $\Psi_i = \int_1^{\xi} \Psi_i'(t)dt + C_i$. For the function $\tilde{\Phi}(\xi, C) = 10^3\Phi(\xi, C)/(1 + 4G^3)$ with $G = \int_1^{\xi} g(t)dt$, where $\Phi(\xi, C)$ is a function as in (8) and (9), we solve the problem of optimization (11) on a uniform net in the interval $[0.2, 1.7]$ with $N = 200$ points. This problem has the solution C_i^* equal to $6.77203 \cdot 10^{-6}$, 0.000127496, 0.00128263, 0.0189312, 0.0367929, -0.0316707, -0.41092, -0.0777118, 1, 0.0289165, -0.12485 for $i = 1, \dots, 11$. The function $\Phi(\xi, C^*)$ is positive for $\xi > 0$, and the equation $\Psi(\xi, \tilde{y}, C^*) = 0$ determines in the region $\xi > 0$ three annuli surrounding the focus $\tilde{A} = (1, 0)$ of system (12). Then, we get the same conclusion than in Example 1. The absence of limit cycles around the focus A follows from works [18, 19].

Example 3: *A quadratic system with* 1 *focus,* 3 *saddles and* 3 *limit cycles surrounding the focus.* We take $a = -2$, $a_{20} = 12$, $a_{11} = 10.999$, $a_{10} = -26.1$ and $a_{01} = -14$. In this case system (2) has the focus A and the three saddles $S_i = (t_i, -1/t_i)$ with $t_1 = -0.67$, $t_2 = 0.15$ and $t_3 = 1.7$. In addition, there are at least 3 limit cycles around the focus A which pass through the points $(x_i, -1)$ with $x_1 = 0.32$, $x_2 = 0.66$ and $x_3 = 0.8$. We show that this system has at most 3 limit cycles. For that purpose we consider the Lienard polynomial system (14) associated to system (2) with

$$F(x) = -\frac{1001}{3000} - 3x + 7x^2 - \frac{10999}{3000}x^3, \quad g(x) = 2x - 14x^2 - 2.1x^3 + 26.1x^4 - 12x^5.$$

The function $\Psi(x, y, C)$ is as in (7) with $n = 10$, $k = -1$ and $\Psi_i = \int_0^{x} \Psi_i'(t)dt + C_i$. For the function $\tilde{\Phi}(x, C) = 100\Phi(x, C)/(1 + 8G^3)$ with $G = \int_1^{x} g(t)dt$, where $\Phi(x, C)$

is a function satisfying (8) and (9), we solve the problem of optimization (11) on a uniform net in the interval $[0.2, 1.72]$ with $N = 650$ points. The problem has the solution C_i^* equal to -0.00891837, -0.0884008, -0.322146, -0.448227, 0.240997, 1, 0.119569, -0.547241, -0.0178445, 0.0269709 for $i = 1, \dots, 10$. The function $\Phi(x, C^*)$ is positive in the interval $(0, 1.705)$. The equation $\Psi(x, y, C^*) = 0$ determines for $x \in I$ three annuli surrounding the focus $\tilde{A} = (1, 0)$ of system (14). The limit cycles of system (14) are located in the strip $t_2 < x < t_3$ of the plane (x, y). The interval I contains the interval (t_2, t_3). Now, the conclusion follows in a similar way to the previous examples.

Example 4: *A quadratic system with* 1 *focus,* 1 *saddle and* 3 *limit cycles surrounding the focus.* We take $a = -2$, $a_{20} = -1$, $a_{11} = 9.49965$, $a_{10} = 6.955$ and $a_{01} = -12.5$. In this case system (2) has the focus A and the saddle $S = (x_0, -1/x_0)$ with $x_0 = 0.2$. In addition, there are at least three limit cycles around the focus A which pass through the points $(x_i, -1)$ with $x_1 = 0.56$, $x_2 = 0.75$ and $x_3 = 0.87$. We consider the Lienard polynomial system (14) associated to system (2) with

$$F(x) = -\frac{782}{9375} - 3x + \frac{25}{4}x^2 - \frac{118747}{375}x^3, \quad g(x) = 2x - \frac{25}{2}x^2 + \frac{3291}{200}x^3 + \frac{1391}{200}x^4 + x^5.$$

Moreover, the function $\Psi(x, y, C)$ is as in (7) with $n = 12$, $k = -1$ and $\Psi_i = \int_0^x \Psi_i'(t)dt + C_i$. For the function $\tilde{\Phi}(x, C) = 10^5\Phi(x, C)/(1 + 4G^2)$ with $G = \int_1^x g(t)dt$, where $\Phi(x, C)$ satisfies (8) and (9), we solve the problem of optimization (11) on a uniform net in the interval $[0.3, 1.4]$ with $N = 750$ points. The problem has the solution C_i^* equal to -0.0257346, -0.141113, -0.371849, -0.602612, -0.602612, -0.281479, 0.102264, 0.157096, 0.0116869, -0.0191466, -0.00362004, 0.000197399 for $i = 1, \dots, 12$. The function $\Phi(x, C^*)$ is positive on the interval $I = (0, 1.8)$, but not on interval $I = (x_0, +\infty)$. The equation $\Phi(x, y, C^*) = 0$ determines for $x \in I$ three ovals. For evaluating the number of limit cycles on the strip $x > x_0$ of the plane (x, y), we shall use the *method of reduction to the global uniqueness of a limit cycle*.

We consider the Andronov–Hopf function $AH(x) = a_{11}$ with $AH(1) = 9.5$ associated to our system (2). We recall that a_{11} is a rotating parameter for system (2). We fix all the parameters and we move only the parameter a_{11}. The function $AH(x)$ is considered on the interval $I_1 = [x_0, x_{max}]$ where the endpoints satisfy $x_0 < 1$ and $x_{max} > 1$, and x_{max} corresponds to the bifurcation of a limit cycle from a loop of the saddle S. If in a subinterval $I_0 = [x_1, x_2]$ of I_1 the number of zeros of the function $AH(x) = a_{11}^0$ is $2p$, then the number of limit cycles of system (2) in the strip $x_1 < x < x_2$ is p. Now, suppose that the equation $AH(x) = a_{11}^1$ with $a_{11}^1 < a_{11}^0$ provides a unique limit cycle which is localized in the strip $x_3 < x < x_4$ with $[x_3, x_4] \subset I_0 \subset I_1$, then the function $AH(x)$ cannot take the value a_{11}^0 outside the interval I_1. Consequently, for the value a_{11}^0 system (2) has exactly p limit cycles. This is the method of reduction to the global uniqueness of a limit cycle.

Now we go back to our particular system (2). Approximately $AH(x)$ is equal to

$$8.89863 + 4.39482x - 13.5991x^2 + 22.9703x^3 - 22.4248x^4 + 11.9886x^5 - 2.72941x^6,$$

on $I_0 = [0.6, 0.9]$. Of course $I_0 \subset I_1$. If we prove, for some a_{11} and remaining fixed the other parameters, that system (14) has a unique limit cycle on a strip $x \in (\check{x}, \hat{x})$ of the plane (x, y) with $(\check{x}, \hat{x}) \subset I_1$, then the function $AH(x)$ does not take the value $a_{11} = 9.49965$ outside the interval I_1, and $AH(x)$ has its complicated behavior only on I.

Now, we take $a_{11} = 9.4993$. Then, system (14) has a limit cycle, which is located on the strip $x_0 < x < 1.8$. We prove its uniqueness. For that we find functions $\Psi(x, y, C)$ and $\Phi(x, C)$ as before with $n = 5$ and $k = -2/3$. The problem (11) has the solution C_i^* equal to -0.126609, -0.0834262, -1, -0.253441, 0.207481 for $i = 1, \ldots, 5$. The function $\Phi(x, C^*)$ is positive for $x > 0$. This means that systems (14), and the corresponding system (2) have for $a_{11} = 9.4993$ a unique limit cycle. Therefore, by applying the method of reduction to the global uniqueness of a limit cycle, the proof of the distribution of 3 limit cycles around the focus A for the considered system (2) follows.

Example 5: *A quadratic system with* 2 *saddles,* 1 *focus,* 1 *node and* 3 *limit cycles surrounding the focus.* We take $a = -4$, $a_{20} = -1$, $a_{11} = 13.9987$, $a_{10} = 12.4$ and $a_{01} = -21$. In this case system (2) has the focus A and the node $N = (t_0, -1/t_0)$ with $t_0 = 9.69$, and two saddles $S_i = (t_i, -1/t_i)$ with $t_1 = 0.29$ and $t_2 = 1.42$. In addition, there are at least three limit cycles which are located on the strip $t_1 < x < t_2$ of the plane (x, y) around the focus A and pass through the points $(x_i, -1)$ with $x_1 = 0.63$, $x_2 = 0.8$ and $x_3 = 0.88$. For computing the number of limit cycles we consider the Lienard polynomial system (14) associated to system (2) with

$$\begin{aligned} F(x) &= -\frac{17539}{150000} - \frac{7}{3}x^3 + \frac{21}{4}x^4 - \frac{139987}{50000}x^5, \\ g(x) &= x^5\left(4 - 21x + \frac{142}{5}x^2 - \frac{62}{5}x^3 + x^4\right), \end{aligned}$$

and we find a function $\Psi(x, y, C)$ as in (7) with $n = 11$, $k = -1$ and $\Psi_i = \int_1^x \Psi_i'(t)dt + C$. Now, for the function $\tilde{\Phi}(x, C) = 10^6\Phi(x, C)/(1 + 4G^2)$ with $G = \int_1^x g(t)dt$, where $\Phi(x, C)$ satisfies (8) and (9), we solve the problem (11) on a uniform net on the interval $[0.5, 1.33]$ with $N = 750$ points. The problem has the solution C_i^* equal to -0.206646, -0.701459, -1, -0.745283, -0.24893, 0.0331453, 0.0341755, 0.000943157, -0.00105898, $-5.55364 \cdot 10^{-6}$, $2.0652\dot{4}10^{-6}$ for $i = 1, \ldots, 11$. The equation $\Psi(x, y, C^*) = 0$ defines in the strip $0.25 < x < 1.3$ only three ovals. The function $\Phi(x, C^*)$ is positive on $(0.25, 1.3)$, but not on $I = (t_1, t_2)$ where limit cycles are located. As in the previous example we can use the method of reduction to the global uniqueness of a limit cycle. We take $a_{11} = 13.998$ and fix the remaining parameters. The corresponding Lienard polynomial system (14) has a limit cycle which is located on the strip $0.25 < x < 1.3$. Then, we find functions $\Psi(x, y, C)$ and $\Phi(x, C)$ as before with $n = 7$, $k = -2/3$ and $a_{11} = 13.998$. The problem (11) has a solution C_i^* equal to -0.884833, -0.874942, -1, -0.158469, -1, -0.107391, 0.0599698 for $i = 1, \ldots, 7$. The function $\Phi(x, C^*)$ is positive for $x > 0$. This means that systems (14), and the corresponding system (2) have for $a_{11} = 13.998$ a unique limit cycle. Now, following with the method of reduction we can complete the proof of the distribution of 3 limit cycles surrounding the focus A of the considered system (2).

Example 6: *A quadratic system with* 1 *saddle,* 1 *focus,* 2 *nodes and* 3 *limit cycles surrounding the focus.* We take $a = 5$, $a_{20} = -50$, $a_{11} = -5.49995$, $a_{10} = 76.45$ and $a_{01} = 16.5$. Then, system (2) has the focus A, the nodes $N_1 = (t_1, -1/t_1)$, $N_2 = (t_2, -1/t_2)$ with $t_1 = -0.46$, $t_1 = 0.34$, and the saddle $S = (t_3, -1/t_3)$ with $t_3 = 0.65$. Also it has at least three limit cycles around the focus A, which pass through the points $(x_i, -1)$ with $x_1 = 1.05$, $x_2 = 1.16$ and $x_3 = 1.5$. For estimating the number of limit cycles we consider the Lienard polynomial system (12) with

$$\begin{aligned} F(\xi) &= -\frac{22003}{240000} - \frac{1099999}{80000}\xi^4 + \frac{33}{10}\xi^5 - \frac{11}{6}\xi^6, \\ g(\xi) &= \xi^7\left(-50 + \frac{1529}{20}\xi - \frac{299}{20}\xi^2 - \frac{33}{2}\xi^3 + 5\xi^4\right). \end{aligned}$$

As before we find functions $\Psi(\xi, \tilde{y}, C)$ and $\Phi(\xi, C)$ satisfying (7), (8) and (9) with $n = 10$, $k = -1$ and $\Psi_i = \int_1^\xi \Psi_i'(t)dt + C_i$ for $i = 1, ..., n$. For the function $\tilde{\Phi}(\xi, C) = 10^3\Phi(\xi, C)/\xi^3)$ we solve the problem (11) on a uniform net in the interval $[0.6, 1.21]$ with $N = 450$ points. For the computations it is better to do the change of variable $\xi \to \xi + 1$. The problem (11) has the solution C_i^* equal to -0.0104019, -0.0613161, -0.329415, -1, 0.0849137, 0.770697, 0.0133268, -0.124194, -0.000345956, 0.00107251 for $i = 1, \ldots, 10$. The equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines in the strip $\xi \in I = (0.1; 1.5)$ only three ovals. The function $\Phi(\xi, C^*)$ is positive on the interval I. Therefore, we use the reduction to a global uniqueness of a limit cycle in the half–plane $\xi > 0$. We take $a_{11} = -5.4997$ and suppose that remaining parameters are fixed. Then, the corresponding system (12) has a limit cycle which is located on the strip $\xi \in I$. Furthermore, we find functions $\Psi(\xi, \tilde{y}, C)$ and $\Phi(\xi, C)$ as before with $n = 5$, $k = -2/3$ and $a_{11} = -5.4997$. The problem (11) has the solution C_i^* equal to -0.029957, -0.00827985, -1, -0.104843, 0.487508 for $i = 1, \ldots, 5$. The function $\Phi(\xi, C^*)$ is positive on $(0, 1.8)$, and the equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines for $0 < \xi < 1.8$ only one oval. Hence, it follows the uniqueness of the limit cycle for the considered system (12). Finally, the original system (2) has exactly three limit cycles in the half–plane $x > 0$ around the focus A.

Example 7: *A quadratic system with* 2 *foci,* 1 *saddle at infinity, and the configuration* $(3, 1)$ *of limit cycles.* We take $a = 8/11$, $a_{20} = -12$, $a_{11} = 2.1502$, $a_{10} = -26.5$ and $a_{01} = 67/220$. The corresponding system (2) has the foci A and $B = (x_0, -1/x_0)$ with $x_0 = -3.2$, and a saddle at infinity. In addition, there are at least three limit cycles around A, which pass through the points $(x_i, -1)$ with $x_1 = 1.28$, $x_2 = 1.15$ and $x_3 = 4.43$; and there is at least one limit cycle around B. For studying the limit cycles surrounding the focus A we consider the associated system (12), which after the change of variable $y = 5\tilde{y}$ has the functions

$$\begin{aligned} F(\xi) &= \frac{10130461}{5700000} - \frac{118261}{75000\xi^{\frac{3}{11}}} + \frac{67}{800}\xi^{\frac{8}{11}} - \frac{7}{95}\xi^{\frac{19}{11}}, \\ g(\xi) &= -\frac{-12}{25\xi^{\frac{17}{11}}} - \frac{53}{5\xi^{\frac{6}{11}}} + \frac{8377}{5500}\xi^{\frac{5}{11}} - \frac{67}{5500}\xi^{\frac{6}{11}} + \frac{8}{275}\xi^{\frac{27}{11}}. \end{aligned}$$

For system (12) we find functions $\Psi(\xi, \tilde{y}, C)$ and $\Phi(\xi, C)$ satisfying (7), (8) and (9) with $n = 11$, $k = -1$ and $\Psi_i = \int_1^\xi \Psi_i'(t)dt + C_i$ for $i = 1, ..., n$. Now, for the function $\tilde{\Phi}(\xi, C) = \Phi(\xi, C)\xi^4/(1 + \xi^9)$ we solve the problem (11) on a uniform net in the interval $[0.001, 4]$ with $N = 790$ points. The problem has the solution C_i^* equal to -0.000309912, -0.00513088, -0.372386, -0.154282, -0.328544, 0.150592, 1, 0.0871286, -0.586201, -0.0121769, 0.0273162 for $i = 1, \ldots, 11$. The equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines in the strip $0 < \xi < 4$ only three ovals. The function $\Phi(\xi, C^*)$ is positive only on $(0, 4)$. Now we use again the method of reduction to the global uniqueness of a limit cycle for proving that there exist exactly three limit cycles surrounding the focus A. We take in system (2) $a_{11} = 2.156$ and the remaining parameters are fixed. Therefore, the corresponding system (12) has a limit cycle which is located on the strip $0 < \xi < 4$ of the phase plane $(\xi, \tilde{y})$. Now we find functions $\Psi(\xi, \tilde{y}, C)$ and $\Phi(\xi, C)$ as before with $n = 3$, $k = -2/3$ and $a_{11} = 2.156$. The corresponding problem (11) has the solution C_i^* equal to -0.8033395, -0.299759, 1 for $i = 1, 2, 3$. The function $\Phi(\xi, C^*)$ is positive for $\xi > 0$. This means that systems (2) and (12) with $a_{11} = 2.156$ have a unique limit cycle, but they for $a_{11} = 2.1502$ have exactly three limit cycles around the focus A. Now, we shall prove the uniqueness of the limit cycle around the focus B for the original system (2) in the half–plane $x < 0$. In fact this uniqueness follows from the results of Zhang Pingguang [20, 21], but here we provide an independent proof. For doing that first, we translate the point B to the point A by means of the change of variables $x = x_0\hat{x}$, $y = \hat{y}/x_0$. System (2) becomes another quadratic system also in the form (2) and its parameters are $\tilde{a}_{00} = x_0^2 a_{00}$, $\tilde{a}_{10} = x_0^3 a_{10}$, $\tilde{a}_{20} = x_0^4 a_{20}$, $\tilde{a}_{01} = x_0 a_{01}$, $\tilde{a}_{11} = x_0^2 a_{11}$ and $\tilde{a} = a$. The uniqueness of the limit cycle is obtained in the half–plane $\tilde{x} > 0$. We can prove this with the help of the functions $\Psi(\xi, \tilde{y}, C)$ and $\Phi(\xi, C)$ satisfying (7), (8) and (9), and the corresponding system (12) with $n = 3$, $k = -1$ and $\Psi_i = \int_1^\xi \Psi_i'(t)dt + C_i$ for $i = 1, 2, 3$. The problem (11) for the function $\tilde{\Phi}(\xi, C) = 10^{-2}\Phi(\xi, C)\xi^{20/11}$ has the solution C_i^* equal to $-8.0101 \cdot 10^{-5}$, $7.10538 \cdot 10^{-4}$, 1 for $i = 1, 2, 3$. The function $\Phi(\xi, C^*)$ is positive for $\xi > 0$, and the equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines for $\xi > 0$ only one oval. By Theorem 3, the considered system has exactly one limit cycle in the half–plane $\xi > 0$, and the original system (2) has exactly one limit cycle around focus B. So, we have distribution $(3, 1)$ of limit cycles for our system (2).

Example 8: *A quadratic system with* 2 *foci, and* 1 *saddle and* 1 *antisaddle at infinity, and the configuration* $(3, 1)$ *of limit cycles.* In [1] the domain of parameters is found in order that a quadratic system has a weak focus of third order and a limit cycle around the other focus. Using these systems we can find a quadratic system (2) with the distribution of limit cycles $(3,1)$. We take $a = 1.04$, $a_{20} = -120$, $a_{11} = 1.51997$, $a_{10} = -79.6$ and $a_{01} = 1.56$. Then, system (2) has the foci A and $B = (x_0, -1/x_0)$ with $x_0 = -1.79$, one saddle and one node at infinity. In addition there are at least three limit cycles around A, which pass through the points $(x_i, -1)$ with $x_1 = 1.29$, $x_2 = 2.22$ and $x_3 = 4.63$; and there is at least one limit cycle around B. For studying the limit cycles surrounding the focus A we consider the associated system (12), which after the change of variable $y = 10\tilde{y}$ has the functions $F(\xi)$, $g(\xi)$:

$$\begin{aligned} F(\xi) &= -\frac{7749847}{2040000} + \frac{151997}{40000}\xi^{\frac{1}{25}} + \frac{3}{20}\xi^{\frac{26}{25}} - \frac{77}{510}\xi^{\frac{51}{25}}, \\ g(\xi) &= -\frac{-6}{5\xi^{\frac{23}{25}}} - \frac{119}{250}\xi^{\frac{2}{25}} + \frac{5003}{2500}\xi^{\frac{27}{25}} - \frac{39}{2500}\xi^{\frac{52}{25}} + \frac{13}{1250}\xi^{\frac{77}{25}}. \end{aligned}$$

As before we find the functions $\Psi(\xi, \tilde{y}, C)$ and $\Phi(\xi, C)$ satisfying (7), (8) and (9) with $n = 10$, $k = -1$ and $\Psi_i = \int_1^\xi \Psi_i'(t)dt + C_i$ for $i = 1, ..., n$. Now, for the function $\tilde{\Phi}(\xi, C) = 10^3\Phi(\xi, C)/(1 + 4G^2)$ with $G = \int_1^\xi g(t)dt$ we solve the problem (11) on a uniform net in the interval $[0.1, 2.2]$ with $N = 400$ points. The problem has the solution C_i^* equal to $9.774 \cdot 10^{-5}$, 0.00294242, 0.035928, 0.273929, -0.0477983, -1, -0.385115, 0.80362, 0.00645912, -0.00449499 for $i = 1, \ldots, 10$. The equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines in the strip $0 < \xi < 5$ only three ovals. The function $\Phi(\xi, C^*)$ is positive on the interval $I = (0; 5)$. Again we use the method of reduction to the global uniqueness of a limit cycle for proving that there exist exactly three limit cycles surrounding the focus A. We take $a_{11} = 1.5198$ and the remaining parameters are fixed. The corresponding system (12) has a limit cycle which is located in the strip $0 < \xi < 5$ of the phase plane $(\xi, \tilde{y})$. Now we find functions $\Psi(\xi, \tilde{y}, C)$ and $\Phi(\xi, C)$ as before with $n = 7$, $k = -1$ and $a_{11} = 1.5198$. The problem (11) has the solution C_i^* equal to 0.00132064, 0.0450009, 1, -0.00941069, 0.20056, 0.134724, -1 for $i = 1, \ldots, 7$. The function $\Phi(\xi, C^*)$ is positive for $\xi > 0$, and the equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines for $\xi > 0$ only one oval. By Theorem 3, the considered system has exactly one limit cycle in a half–plane $\xi > 0$, then the original system (2) has exactly three limit cycles in a half–plane $x > 0$ around the focus A. The uniqueness of the limit cycle around B for the original system (2) in the half–plane $x < 0$ is proved in the same way as in Example 7 if the function $\tilde{\Phi}(\xi, C^*)$ is equal to $\Phi(\xi, C)\xi^{24/25}/10^6$. The problem (11) has the solution C_i^* equal to $1.24736 \cdot 10^{-4}$, 0.00948753, 1 for $i = 1, 2, 3$. The function $\Phi(\xi, C^*)$ is positive for $\xi > 0$, and the equation $\Psi(\xi, \tilde{y}, C^*) = 0$ defines for $\xi > 0$ only one oval, which allows to show the uniqueness of the limit cycle of system (2) around the focus B.

Remark 1. *Kooij and Zegeling proved in [18, 19] that the distribution of limit cycles* $(3, 1)$ *is possible only for quadratic system of the type* $2A + 1S_\infty$, $2A + 2S_\infty + 1S_\infty$ *which we have considered.*

Remark 2. *For constructing the examples of quadratic system with the maximum number of limit cycles it is not necessary to use the function* $\tilde{a}_{11}(x)$*. It is enough to know that the function* $a_{10}(x)$ *has an extremum, then the function* $AH(x)$ *will have two extrema and provides the existence of an interval for the function* a_{11} *in which the system has three limit cycles. Also it is possible instead of using the normal form given by system (2), to use other canonical families of quadratic systems considered in [9, 17].*

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