BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(41), 2003, Pages 3–6
ISSN 1024–7696

Constantin
Sergeevich
Sibirsky
(1928 – 1990)

*This issue is
a tribute in honor
of his 75th birthday*

The mathematical legacy of C.S. Sibirsky, basis for future work

Dana Schlomiuk
Département de Mathématiques et de Statistiques
Université de Montréal

In 1985 while spending the fall term at Cornell University, part of my sabbatical leave from the Université de Montréal, I worked with John Guckenheimer and Richard Rand on quadratic perturbations of quadratic Hamiltonian vector fields. For this we needed the conditions for a center for quadratic differential systems and Carmen Chicone provided us with several references. To our surprise we were faced with three sets of conditions which did not seem to be equivalent and indeed, after giving counterexamples in all directions, we had a proof that the three sets produced distinct families of systems. Was there a correct one among the three? There was: C.S. Sibirsky's set of conditions was correct ([8],[9]). This was my first encounter with his work.

As I became more interested in the problem of the center I got to know more of his work. This problem, open for planar polynomial vector fields in any degree greater than two, was solved by Dulac [3] for quadratic systems in 1908. Nevertheless, a compact set of conditions for a center readily applicable to real systems is not what one finds in Dulac's paper. The fourteen or so conditions of Dulac are scattered throughout his paper, interspersed with case by case discussions and his initial canonical form is for systems with a saddle point. Dulac's definition of a center is very general [2] and it is stated for systems which are real or complex: a center is a singular point with nonzero eigenvalues whose quotient is negative and rational and the system has a local analytic first integral around the singular point. Dulac had much foresight in stating it this way. But again, a compact set of conditions for a center, readily applicable to real systems, is very useful and it is also a basis for finding the global implications of the presence of center. As I became more involved with the problem and with the geometrical meaning of the algebraic conditions for a center in the nondegenerate case, I began to exchange letters with C. S. Sibirsky and also our papers.

In 1990 I started to organize the NATO sponsored Advanced Study Institute on "Bifurcations and Periodic Orbits of Vector Fields" which was to be held in Montreal in July 1992. Naturally I wished that C. S. Sibirsky would be one of the speakers and so I sent him a letter asking him to be a main speaker at this Institute. The reply to my letter failed to arrive. Instead I got the very sad news of his passing away. The loss must have been overwhelming for his family, for his former students, for the Chişinău mathematical community. For people working on planar polynomial vector fields elsewhere in the world, a field in which he made substantial contributions and was a leader, the loss was also great. He left solid work and the school he built was to survive and continue under difficult conditions. Indeed, now twelve years after his death, his work is being vigorously continued by his former students and their own students and their work is beginning to receive fully deserved international recognition.

C.S. Sibirsky's research was in the area of differential equations and dynamical systems. Along with his other interests (for example cf. [10]), part of K.S. Sibirkii's contributions to mathematics were devoted to developing the invariant theory of differential equations [11]–[13]. Initially, the theory of algebraic invariants was developed for n-forms in m variables. The invariant property of the discriminant of two-forms with respect to special linear transformations was noticed by Gauss in 1801 and in 1841 the treatise of Boole [1] launched the study of algebraic invariants of n-forms. Work by British, German, and later French and Italian mathematicians contributed to developing this theory. In the 1890's Hilbert solved the two main problems of invariant theory of n-ary forms [4], [5]. Hermann Weyl [14, p. 27] had this to say about the work of Hilbert: "Here there is only one man to mention – Hilbert. His papers (1890/92) mark a turning point in the history of invariant theory. He solves the problem and thus almost kills the subject". Not only did the invariant theory survive and further evolve but new interesting problems were stated. In its modern day version, formulated by Mumford [7] as geometric invariant theory, it connects with present day main stream mathematics in the form of moduli theory. The language used is the language of algebraic geometry, a subject with close connections with invariant theory from its beginning. It is interesting to view C. S. Sibirsky's work on invariant theory of polynomial differential equations from this wider perspective as it shows that his work evolved in a very well oriented direction with much potential for future development.

C. S. Sibirsky constructed algebraic invariants and comitants for polynomial vector fields. These were used for the purpose of classifying such systems, for example classifying (under topological equivalence) specific families of quadratic or cubic differential systems. Many articles on classifying such systems, containing nonintrinsic classifications, were published elsewhere in the world. To be specific, these classifications were given in terms of inequalities involving the coefficients of specific normal forms for the families involved. They are not readily applicable to other presentations of the systems.

C. S. Sibirsky was concerned with making the classifications intrinsic, i.e. invariant under allowable coordinate changes and hence independent of the specific presentation of the systems. And he and his former students were successful in giving intrinsic classifications for a number of families of polynomial systems, which were achieved due to the use of algebraic invariants and comitants.

The task of finding algebraic invariants transparent enough to let the geometry of the systems and even of their respective families filter through was left undone. It is in this direction that much work remains to be achieved and his former students pursue this goal. In the case of algebraic families of systems such as, for example, the class of quadratic systems with center, the classification is completely done in terms of purely geometric properties of the systems and can be described by inequalities expressed in terms of algebraic invariants. This is a family of integrable systems. When we need to classify families of nonintegrable differential systems, inequalities expressed in terms of algebraic invariants do not suffice but they necessarily appear in the initial part of such classifications. In general the study of a specific family of polynomial systems, for example the quadratic family, implies the use of a number of charts which eventually need to be glued together to produce the whole picture. We therefore need to use distinct presentations of the systems and be able to easily pass from one to another. The algebraic invariants and comitants are of help here. Although they could not do the whole work (classifying nonintegrable systems implies the use of not only algebraic inequalities but also analytic or smooth ones and a lot of analytic work along with some numerical analysis) they are a necessary part of the work. Furthermore, in a number of problems of an algebraic or algebro-geometric nature, the theory of algebraic invariants and comitants of differential equations intervenes substantially. Polynomial differential systems with specific algebraic properties, for example those possessing algebraic invariant curves, are interesting in their own right and have potential for applications. A specific example is provided by the study of quadratic differential systems with a third order focus: the bifurcation diagram (cf. [6]) of this family is basically obtained by using perturbations of systems with a center, an algebraic family of systems possessing invariant algebraic curves. Work remains to be done and the general theory of planar polynomial systems whether they be equipped with additional algebraic properties or not is in the process of growing and some of its connections with algebraic geometry are emerging. Much work in this field was done from the viewpoint of bifurcation theory. While bifurcation theory is one of the basic ingredients in this area, confining the study to bifurcation theory only, is a limitation as other viewpoints help in understanding the subject and strengthen the possibility of bringing solutions to its problems.

One of the precious qualities of C. S. Sibirsky's work lies in his approach using a different viewpoint and its connections extend to present day geometric invariant theory. A lot remains to be done and work by the people he so well trained, their students and their collaborators, is expanding in this direction. On the 75th anniversary of his birth his work is very much present on the mathematical scene and new offshoots arise from his mathematical legacy. The people he formed and scientists elsewhere in the world pursue his vision and the problems which fascinated him, as well as newly formulated ones, may one day be solved.

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Département de Mathématiques et de Statistique,
Université de Montréal,
C. P. 6128, Succursale Centre-Ville,
Montréal, (Québec) H2V 3C5 Canada
e-mai: dasch@dms.umontreal.ca

Received December 10, 2002