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A Lie algebra of a differential generalized FitzHugh–Nagumo system

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Abstract. Some Lie algebra admissible for a generalized FitzHugh-Nagumo (F-N) system is constructed. Then this algebra is used to classify the dimension of the $Aff_3(2, R)$-orbits and to derive the four canonical systems corresponding to orbits of dimension equal to 1 or 2. The phase dynamics generated by these systems is studied and is found to differ qualitatively from the dynamics generated by the classical F-N system the $Aff_3(2, R)$-orbits of which are of dimension 3. A dynamic bifurcation diagram is also presented.

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1 The Lie algebra admissible for the generalized FitzHugh–Nagumo system

We investigate the generalized F-N system [1]

$$\dot{x} = a + cx + dy + px^3 \equiv P(x, y), \quad \dot{y} = b + ex + fy \equiv Q(x, y), \tag{1}$$

where the coefficients a, b, c, d, e, f, p are real and the phase functions x and y are real-valued and depend on the real time t; the dot over quantities stands for d/dt.

By [2], the Lie algebra consisting of the operators

$$X = \xi^1(x, y)\frac{\partial}{\partial x} + \xi^2(x, y)\frac{\partial}{\partial y} + D,$$

$$D = \eta^1\frac{\partial}{\partial a} + \eta^2\frac{\partial}{\partial b} + \eta^3\frac{\partial}{\partial c} + \eta^4\frac{\partial}{\partial d} + \eta^5\frac{\partial}{\partial e} + \eta^6\frac{\partial}{\partial f} + \eta^7\frac{\partial}{\partial p} \tag{2}$$

is admissible for (1) iff the coordinates of these operators satisfy the following system of partial differential equations

$$\xi^1_x P + \xi^1_y Q = \xi^1 P_x + \xi^2 P_y + DP,$$

$$\xi^2_x P + \xi^2_y Q = \xi^1 Q_x + \xi^2 Q_y + DQ, \tag{3}$$

where the coordinates ξ^1, ξ^2 and $\eta^i (i = \overline{1, 7})$ are unknown functions of x and y and of the coefficients a, b, c, d, e, f, p, respectively.

Let us assume that ξ^1, ξ^2 are affine while η^i are linear functions, i.e.

$$\xi^1 = A + Bx + Cy, \quad \xi^2 = H + Kx + Ly,$$
$$\eta^i = \alpha_1^i a + \alpha_2^i b + \alpha_3^i c + \alpha_4^i d + \alpha_5^i e + \alpha_6^i f + \alpha_7^i p \quad (i = \overline{1,7}). \tag{4}$$

In this way, the determination of the unknown functions is reduced to the solution of an algebraic system in A, B, C, H, K, L, $\alpha_j^i (i, j = \overline{1,7})$. It is found

$$X = B\left(x\frac{\partial}{\partial x} + a\frac{\partial}{\partial a} + d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e} - 2p\frac{\partial}{\partial p}\right) +$$
$$+H\left(\frac{\partial}{\partial y} - d\frac{\partial}{\partial a} - f\frac{\partial}{\partial b}\right) +$$
$$+L\left(y\frac{\partial}{\partial y} + b\frac{\partial}{\partial b} - d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e}\right). \tag{5}$$

Since B, H and L are arbitrary, the expression (5) represents a family of operators, among which the operators corresponding to 1)$B = 0$, $H = 0$, $L = -1$; 2)$B = 0$, $H = -1$, $L = 0$; 3)$B = -1$, $H_3 = 0$, $L_3 = 0$ play a special role. Namely, we have

Theorem 1. *The maximum number of linearly independent Lie operators admissible for (1) and the coordinates of which are affine and linear functions of the phase functions and the parameters in (1), respectively, is equal to 3, and a triple of these operators reads*

$$X_1 = -y\frac{\partial}{\partial y} - b\frac{\partial}{\partial b} + d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e},$$
$$X_2 = -\frac{\partial}{\partial y} + d\frac{\partial}{\partial a} + f\frac{\partial}{\partial b},$$
$$X_3 = -x\frac{\partial}{\partial x} - a\frac{\partial}{\partial a} - d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e} + 2p\frac{\partial}{\partial p}. \tag{6}$$

It can be checked immediately that

Remark 1. *The Lie operators (6) form a three-dimensional Lie algebra the commutators of which are given in Table 1.*

Table 1

	X_1	X_2	X_3
X_1	0	X_2	0
X_2	$-X_2$	0	0
X_3	0	0	0

Denote this algebra by L_3.

Remark 2. *The Lie algebra L_3 is resolvable and has the operator X_3 as a nonnull central element.*

2 Functional basis of comitants and invariants

Let $Aff_3(2,R)$ be the group defined by the transformations q:

$$\overline{x} = \alpha x, \quad \overline{y} = \beta y + h, \quad \Delta = \alpha\beta \neq 0, \tag{7}$$

where $\alpha, \beta, h \in R$.

Lemma 1. *Performing in (1) the transformations (7) we get the system*

$$\dot{\overline{x}} = \overline{a} + \overline{cx} + \overline{dy} + \overline{px}^3, \quad \dot{\overline{y}} = \overline{b} + \overline{ex} + \overline{fy},$$

where

$$\overline{a} = \alpha a - \frac{\alpha hd}{\beta}, \quad \overline{c} = c, \quad \overline{d} = \frac{\alpha d}{\beta}, \quad \overline{p} = \frac{p}{\alpha^2},$$
$$\overline{b} = \beta b - hf, \quad \overline{e} = \frac{\beta e}{\alpha}, \quad \overline{f} = f. \tag{8}$$

By [2], by solving the Lie equations for the operators (6) we have

Remark 3. *The Lie algebra L_3 is equivalent to the $Aff_3(2,R)$-group affine representation by formulae (7) in the space of coefficients of the system (1) given by (8).*

From Theorem 1 and Remarks 1 and 3 we obtain

Corollary 1. *The largest affine group admissible for system (1) is $Aff_3(2,R)$ defined by formulae (7).*

Definition 1. *A polynomial $k(x,y,a,b,c,d,e,f,p)$ of the coefficients of the system (1) and variables x and y is called a comitant of this system with respect to the $Aff_3(2,R)$-group if the identity*

$$k(\overline{x},\overline{y},\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},\overline{p}) = \Delta^{-g} k(x,y,a,b,c,d,e,f,p) \tag{9}$$

holds for every coefficients and variables of the system (1) and every parameters α,β,$h \in R$ of the $Aff_3(2,R)$-group.

If the comitant k does not depend on the variables x and y then it is referred to as the invariant of the system (1) with respect to the $Aff_3(2,R)$-group and is denoted by j. The integer g in (9) is called the $k(j)$ comitant (invariant) weight. If $g \neq 0$ then the comitant (invariant) is said to be relative and otherwise it is absolute.

Let us introduce the following operators from (6)

$$D_1 = -b\frac{\partial}{\partial b} + d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e}, \quad D_2 = d\frac{\partial}{\partial a} + f\frac{\partial}{\partial b},$$
$$D_3 = -a\frac{\partial}{\partial a} - d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e} + 2p\frac{\partial}{\partial p}. \tag{10}$$

Theorem 2. *The polynomial $k(j)$ is the comitant (invariant) of the system (1) of weight g with respect to the $Aff_3(2,R)$ -group iff the equalities*

$$X_1(k) = X_3(k) = gk, \quad X_2(k) = 0$$

$$(D_1(j) = D_3(j) = gj, \quad D_2(j) = 0) \tag{11}$$

hold, where $X_1 - X_3$ are given by (6) while $D_1 - D_3$ by (10).

Proof. Let us examine the operators $\alpha\dfrac{\partial}{\partial\alpha}$, $\dfrac{\partial}{\partial h}$, $\beta\dfrac{\partial}{\partial\beta}$.

For them, from (7) we have

$$\alpha\frac{\partial\Delta}{\partial\alpha} = \Delta, \quad \frac{\partial\Delta}{\partial h} = 0, \quad \beta\frac{\partial\Delta}{\partial\beta} = \Delta. \tag{12}$$

Applying the operator $\alpha\frac{\partial}{\partial\alpha}$ to (9) and taking into account (12) we obtain

$$\alpha\frac{\partial}{\partial\alpha}\left[k(\overline{x},\overline{y},\overline{a},\overline{b},\overline{c},\overline{d},\overline{e},\overline{f},\overline{p})\right] = -g\Delta^{-g}k(x,y,a,b,c,d,e,f,p).$$

Differentiating the left-hand side of this equality as a compound function of α, we get

$$\frac{\partial k}{\partial\overline{x}}\left(\alpha\frac{\partial\overline{x}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{y}}\left(\alpha\frac{\partial\overline{y}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{a}}\left(\alpha\frac{\partial\overline{a}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{b}}\left(\alpha\frac{\partial\overline{b}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{c}}\left(\alpha\frac{\partial\overline{c}}{\partial\alpha}\right) +$$

$$+\frac{\partial k}{\partial\overline{d}}\left(\alpha\frac{\partial\overline{d}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{e}}\left(\alpha\frac{\partial\overline{e}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{f}}\left(\alpha\frac{\partial\overline{f}}{\partial\alpha}\right) + \frac{\partial k}{\partial\overline{p}}\left(\alpha\frac{\partial\overline{p}}{\partial\alpha}\right) = -g\Delta^{-g}k.$$

Taking into account (7) and (8) this equality implies

$$\frac{\partial k}{\partial\overline{x}}(\alpha x) + \frac{\partial k}{\partial\overline{a}}\left(\alpha a - \frac{\alpha h d}{\beta}\right) + \frac{\partial k}{\partial\overline{d}}\left(\frac{\alpha d}{\beta}\right) + \frac{\partial k}{\partial\overline{e}}\left(-\frac{\beta e}{\alpha}\right) +$$

$$+\frac{\partial k}{\partial\overline{p}}\left(-\frac{2p}{\alpha^2}\right) = -g\Delta^{-g}k. \tag{13}$$

This identity holds for every transformation (7) of the $Aff_3 3(2,R)$-group. In particular, the equality (13) holds also for the identity transformation given by $\alpha = \beta = 1$, $h = 0$. In this case (13) implies

$$x\frac{\partial k}{\partial x} + a\frac{\partial k}{\partial a} + d\frac{\partial k}{\partial d} - e\frac{\partial k}{\partial e} - 2p\frac{\partial k}{\partial p} = -gk,$$

whence $X_3(k) = gk$.

The other equations in (11) are proved in a similar way. Since the conclusions are invertible, Theorem 2 follows.

Definition 2. *A functional basis of the set of comitants of the system (1) with respect to the* $Aff_3(2,R)$*-group is the set of functionally independent invariants*

$$j_1, j_2, ..., j_m \tag{14}$$

and comitants

$$k_1, k_2, ..., k_n \tag{15}$$

such that every comitant of the system (1) with respect to the given group can be expressed as a function of the elements of (14), (15).

The functional basis of the set of invariants for the system (1) with respect to the $Aff_3(2,R)$-group is defined analogously.The relationships between relative and absolute comitants can be taken into account to prove, by using (11), the validity of the following result.

Theorem 3. *The number of elements of the functional basis of comitants (invariants) of the system (1) with respect to the elements of the* $Aff_3(2,R)$*-group is equal to* ***7(5)****.*

Theorems 2 and 3 can be used to prove

Theorem 4. *The functional basis of the invariants of the system (1) with respect to the* $Aff_3(2,R)$*-group consists of the elements*

$$j_1 = c \ (g=0), \quad j_2 = f \ (g=0), \quad j_3 = dp \ (g=1),$$
$$j_4 = p(af-bd)^2 \ (g=0), \quad j_5 = de \ (g=0), \tag{16}$$

where g *are the corresponding weights of the invariants* j_l $(l = \overline{1,5})$.

Proof. By Theorem 2 the relations $D_i(j_l) = 0 \quad (i = 1,2,3; l = 1,2,4,5)$ and $D_1(j_3) = D_3(j_3) = j_3, \quad D_2(j_3) = 0$ hold, which shows that the expressions (16) are the invariants of the system (1) with respect to the $Aff_3(2,R)$-group. On the other hand, by Theorem 3, these 5 invariants of (16) could form a functional basis for the system (1) with respect to this group. In order to prove the last assertion it is sufficient to show that the general rank of the Jacobi matrix constructed by means of the invariants of (16) is equal to 5.

Remark that the minor constructed on the last 5 columns of this matrix has the form

$$M = det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & p & 0 & 0 & d \\ 0 & -2bp(af-bd) & 0 & 2ap(af-bd) & (af-bd)^2 \\ 0 & e & d & 0 & 0 \end{pmatrix},$$

whence $M = -dp(a^2f^2 - b^2d^2) \not\equiv 0$. The last inequality proves the assertion in Theorem 4.

Theorem 5. *The functional basis of comitants of the system (1) with respect to the $Aff_3(2,R)$-group consists of the elements*

$$j_1 = c\ (g = 0),\quad j_2 = f\ (g = 0),\quad j_3 = dp\ (g = 1),\quad j_4 = p(af - bd)^2\ (g = 0),$$

$$p_1 = px^2\ (g = 0),\quad p_2 = ex^2\ (g = -1),\quad p_3 = bx + fxy\ (g = -1), \tag{17}$$

where g are the corresponding weights of the invariants j_l $(l = \overline{1,4})$ and comitants p_i $(i = \overline{1,3})$.

The proof of Theorem 5 is analogous to the proof of Theorem 4, where as the minor of the Jacobi matrix constructed on the functions (17) is taken the minor situated on its last 7 columns and which is written as $M = 2fp^2(a^2f^2 - b^2d^2)x^4 \not\equiv 0$. This shows that the expressions in (17) form the functional basis of the comitants of the system (11) with respect to the $Aff_3(2,R)$-group. This concludes the proof.

3 Dimension of the $Aff_3(2,R)$-orbits for $p_1 \not\equiv 0$

If $p_1 \equiv 0$ then p=0 and the system (1) becomes

$$\frac{dx}{dt} = a + cx + dy,\quad \frac{dy}{dt} = b + ex + fy$$

and admits the group $Aff_6(2,R)$, which needs a separate investigation.

Let $A = (a,b,c,d,e,f,p) \in E(A)$, where $E(A)$ is the Euclidean space of coefficients in (1). Denote by $A(q)$ the point of $E(A)$ corresponding to the system obtained from (1) by means of the transformation $q \in Aff_3(2,R)$ given by (7).

Definition 3. *The set $O(A) = \{A(q);\, q \in Aff_3(2,R)\}$ is referred to as the $Aff_3(2,R)$-orbit of the point A for the system (1).*

Let M_1 be the matrix the entries of which are the coordinates of operators (10), i.e.

$$M_1 = \begin{pmatrix} 0 & -b & 0 & d & -e & 0 & 0 \\ d & f & 0 & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & -d & e & 0 & 2p \end{pmatrix}. \tag{18}$$

Remark 4. *In [3] it is proved that*

$$dim_R O(a) = rank M_1. \tag{19}$$

Lemma 2. *For $p_1 \not\equiv 0$ the $rank M_1 = 3$ iff*

$$j_2p_2 + j_3 \not\equiv 0. \tag{20}$$

Proof. Denote by Δ_{ijk} $(1 \le i, j, k \le 7)$ the third order minors of the matrix M_1 corresponding to the columns i, j, k. In this case the only possible nonnull such minors are

$$\Delta_{124} = d(af - bd), \quad \Delta_{125} = -e(af - bd), \quad \Delta_{127} = 2bdp,$$

$$\Delta_{147} = -2d^2p, \quad \Delta_{157} = 2dep, \quad \Delta_{247} = -2dfp, \quad \Delta_{257} = 2efp. \tag{21}$$

Therefore $rankM_1 = 3$ iff $d^2 + e^2f^2 \neq 0$. In this case $(\Delta_{147})^2 + (\Delta_{257})^2 \neq 0$, therefore $rankM_1 = 3$. Taking into account that $j_2p_2 + j_3 = efx^2 + dp$, Lemma 2 follows by replacing the conditions on d, e, f by those on $j_2p_2 + j_3$.

Relation (21) shows that

$$j_2p_2 + j_3 \equiv 0 \tag{22}$$

iff

$$d = ef = 0. \tag{23}$$

Whence we have

Corollary 2. *For $p_1 \not\equiv 0$ $rank\, M_1 < 3$ iff (23) holds.*

Lemma 3. *For $p_1 \not\equiv 0$ $rank\, M_1 = 2$ iff*

$$j_2p_2 + j_3 \equiv 0, \quad p_2 + p_3 \not\equiv 0. \tag{24}$$

Proof. Let Δ^{ij}_{kl} $(1 \le i, j \le 3; 1 \le k, l \le 7)$ denote the second order minors of M_1 corresponding to the rows i, j and columns k, l. By (23) we must consider only the following cases: 1) $d = e = f = 0$, 2) $d = e = 0, f \neq 0$, 3) $d = f = 0, e \neq 0$. In the case 1) the only nonnull minor is $\Delta^{13}_{27} = -2bp$, hence $rank\, M_1 = 2$ iff $b \neq 0$, or equivalently, iff $p_2 + p_3 \neq 0$ because $p_2 + p_3 = bx$. In the cases 2) and 3) we have $\Delta^{23}_{27} = 2fp \neq 0$ and $\Delta^{13}_{57} = -2ep \neq 0$, hence $rank\, M_1 = 2$. Since in these cases $p_2 + p_3 \not\equiv 0$, Lemma 3 follows.

It is immediate that the conditions

$$\Delta^{13}_{12} = -ae, \quad \Delta^{13}_{25} = -be, \quad \Delta^{13}_{27} = -2bp,$$

$$\Delta^{13}_{57} = -2ep, \quad \Delta^{23}_{12} = af, \quad \Delta^{23}_{27} = 2fp \tag{25}$$

hold iff

$$p_2 + p_3 \equiv 0. \tag{26}$$

Corollary 3. *If (26) holds then $j_2 = 0$, $p_2 \equiv 0$ holds, too.*

Lemma 4. *For $p_1 \not\equiv 0$ $rankM_1 = 1$ iff*

$$j_3 + p_2 + p_3 \equiv 0. \tag{27}$$

Proof. Lemma 3 shows that $rank\, M_1 < 2$ corresponds to

$$j_2 p_2 + j_3 = 0, \quad p_2 + p_3 \equiv 0, \tag{28}$$

or, equivalently,

$$b = d = e = f = 0. \tag{29}$$

For (29) M_1 becomes

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 & 0 & 2p \end{pmatrix}.$$

Since $p_1 = px^2 \not\equiv 0$ it follows that $rank M_1 = 1$, whence Lemma 4 holds.

The Lemmas 2–4 and relation (19) imply

Theorem 6. *For $p_1 \not\equiv 0$ the $Aff_3(2,R)$- orbits of the system (1) have the following dimensions*

$$3 \quad for \quad j_2p_2 + j_3 \not\equiv 0;$$

$$2 \quad for \quad j_2p_2 + j_3 \equiv 0, p_2 + p_3 \not\equiv 0;$$

$$1 \quad for \quad j_3 + p_2 + p_3 \equiv 0,$$

where j_2, j_3, p_2, p_3 are given by (17).

Definition 4. *The set $N \subseteq E(A)$ is called an $Aff_3(2,R)$-invariant if for every $A \in N$ we have $O(A) \subseteq N$.*

Let us denote by $N_1 \equiv N_1\,(j_3+p_2+p_3 \equiv 0)$ and $N_2 \equiv N_2\,(j_2p_2+j_3 \equiv 0, p_2+p_3 \not\equiv 0)$ the $Aff_3(2,R)$-invariant sets of Theorem 6 the orbits of which have the dimension 1 and 2, respectively. Let us remark that $N_2 = N_2' \bigcup N_2'' \bigcup N_2'''$, where $N_2' \equiv N_2'\,(j_2 = j_3 = 0, p_2 \equiv 0)$, $N_2'' \equiv N_2''\,(j_2 = j_3 = 0, p_2 \not\equiv 0)$, $N_2''' \equiv N_2'''\,(j_2 \neq 0, j_3 = 0,\, p_2 \equiv 0)$ are sets invariant with respect to the $Aff3(2,R)$-group. They are also mutually disjoint.

Remark 5. *The generalized F-N system (1) on the $Aff_3(2,R)$-invariant sets N_1 and N_2 have the following canonical forms*

$$\dot{x} = a + cx + px^3, \dot{y} = 0 \quad on \quad N_1,\ where\ p \neq 0; \tag{30}$$

$$\dot{x} = a + cx + px^3, \dot{y} = b \quad on \quad N_2',\ where\ pb \neq 0; \tag{31}$$

$$\dot{x} = a + cx + px^3, \dot{y} = b + ex \quad on \quad N_2'',\ where\ pe \neq 0; \tag{32}$$

$$\dot{x} = a + cx + px^3, \dot{y} = b + fy \quad on \quad N_2''',\ where\ pf \neq 0. \tag{33}$$

4 Phase dynamics for systems (30)-(33)

For an easier treatment we reduce the number of the parameters by the time rescaling $t = \dfrac{\tau}{p}$. Considering the new parameters

$$r = \frac{a}{p}, \quad q = \frac{c}{p}, \quad m = \frac{b}{p}, \quad n = \frac{e}{p}, \quad s = \frac{f}{p}, \tag{34}$$

systems (30)–(33) become

$$\dot{x} = r + qx + x^3, \quad \dot{y} = 0; \tag{35}$$

$$\dot{x} = r + qx + x^3, \quad \dot{y} = m, \; m \neq 0; \tag{36}$$

$$\dot{x} = r + qx + x^3, \quad \dot{y} = m + nx, \; n \neq 0; \tag{37}$$

$$\dot{x} = r + qx + x^3, \quad \dot{y} = m + sy, \; s \neq 0, \tag{38}$$

where the dot stands for the differentiation with respect to the new time τ and $x \neq 0$. The equilibrium points of these systems satisfy

$$\dot{x} = 0, \quad \dot{y} = 0. \tag{39}$$

That is why we consider first the equation

$$r + qx + x^3 = 0. \tag{40}$$

Its discriminant is $D = \left(\frac{q}{3}\right)^3 + \left(\frac{r}{2}\right)^2$.

Equation (40) has a single real solution x_0 if $D > 0$, three distinct real solutions x_1, x_2, x_3 if $D < 0$ and two distinct real solutions, one of them being double, if $D = 0$. It is convenient to consider the following expressions for these solutions [4]:
a) $D > 0$

a1) if $q = 0$, then

$$x_0 = \sqrt[3]{-r}; \tag{41}$$

a2) if $q > 0$, then

$$x_0 = -2\sqrt{\frac{q}{3}} \sinh\theta, \theta = \frac{1}{3} sinh^{-1}\left(\frac{r}{2\sqrt{\left(\frac{q}{3}\right)^3}}\right); \tag{42}$$

a3) if $q < 0$, then

$$x_0 = -2\sqrt{\frac{q}{3}} \frac{1}{sin(2\phi)}, \quad \phi \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \tag{43}$$

with $tg\,\phi = \sqrt[3]{tg\frac{\psi}{2}}, \quad sin\,\psi = \frac{2}{r}\sqrt{-\left(\frac{q}{3}\right)^3}, \quad \psi \in (-\frac{\pi}{2}, \frac{\pi}{2});$

b) $D < 0$ (so $q < 0$)

$$x_1 = -2\sqrt{-\frac{q}{3}}sin(\Phi + \frac{\pi}{3}), \tag{44}$$

$$x_2 = 2\sqrt{-\frac{q}{3}}sin\Phi, \tag{45}$$

$$x_3 = 2\sqrt{-\frac{q}{3}}sin(\frac{\pi}{3} - \Phi), \tag{46}$$

where $\Phi = \frac{1}{3}sin^{-1}\frac{r}{2\sqrt{\left(-\frac{q}{3}\right)^3}} \in (-\frac{\pi}{6}, \frac{\pi}{6})$. In addition, $x_1 < x_2 < x_3$.

c) $D = 0$ (so $q \leq 0$)

$$x_1 = -2\sqrt{-\frac{q}{3}}, \quad x_2 = x_3 = \sqrt{-\frac{q}{3}} \tag{47}$$

or

$$x_1 = x_2 = -\sqrt{-\frac{q}{3}}, \quad x_3 = 2\sqrt{-\frac{q}{3}}. \tag{48}$$

If $q = 0$ and $D = 0$, then $r = 0$ and $x_1 = x_2 = x_3 = 0$, but this situation will not be considered because $x \neq 0$.

In order to obtain the equilibria of system (35), system (39) must be solved. As the second equation (39) is always satisfied, system (35) has an infinity of equilibria, situated on one, three or two straight lines $x = x_i$ in the phase plane (x, y), as $D > 0$, $D < 0$ or $D = 0$, respectively. The matrix of the linearized system around an equilibrium point (x_i, k) is

$$A_1 = \begin{pmatrix} q + 3x_i^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and the corresponding eigenvalues are $\lambda_1 = 0, \lambda_2 = q + 3x_i^2$. Although all equilibria are nonhyperbolic, their type can be deduced very easy, because the dynamics takes place on straight lines $y = k$. Thus, analyzing the variation of the function $F(x) = r + qx + x^3$, its sign can be found. It follows that x is increasing on the straight lines $y = k$ when $F > 0$ and is decreasing when $F < 0$. That is why, when $D > 0$, the equilibria (x_0, k) are repulsors, when $D < 0$, the equilibria (x_1, k) and (x_3, k) are repulsors and the equilibria (x_2, k) are attractors, while when $D = 0$, the simple equilibrium points are repulsors and the double equilibrium points are degenerated saddles. The bifurcation diagram of system (35) is given in Figure 1.

System (36) has no equilibrium points because the second equation (39) is never satisfied. With the transformation $\frac{y}{m} = y_1$, system (36) becomes

$$\dot{x} = r + qx + x^3, \quad \dot{y} = 1$$

and the equations of the phase trajectories can be found. Thus, if $D < 0$, then

$$\frac{dx}{dy_1} = (x - x_1)(x - x_2)(x - x_3),$$

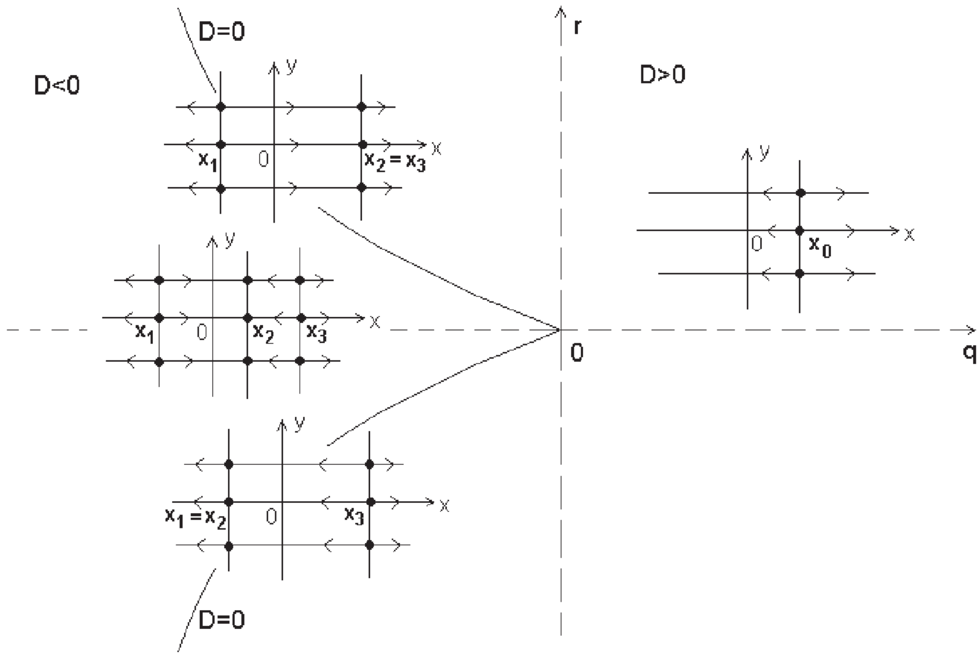

Fig. 1 Bifurcation diagram for system (35)

and the trajectory through (x_{10}, y_{10}) has the equation

$$y_1 = y_{10} + \alpha_1 \ln \left| \frac{x - x_1}{x_{10} - x_1} \right| + \alpha_2 \ln \left| \frac{x - x_2}{x_{10} - x_2} \right| + \alpha_3 \ln \left| \frac{x - x_3}{x_{10} - x_3} \right|$$

where

$$\alpha_1 = \frac{1}{(x_1 - x_2)(x_1 - x_3)}, \ \alpha_2 = \frac{1}{(x_2 - x_1)(x_2 - x_3)}, \ \alpha_3 = \frac{1}{(x_3 - x_1)(x_3 - x_2)}$$

and $y_1 = \tau + y_{10}$.

Similar formulas for $D \geq 0$ can be obtained.

The second equation (39) for system (37) gives $x = -\frac{m}{n}$. Consequently, system (37) has no equilibrium points if $-\frac{m}{n} \neq x_i$, $i = 0, 1, 2, 3$, where the expressions for x_i are given by (41)–(48) and it has an infinity of equilibria situated on the straightline $x = x_i$ of the phase plane if $-\frac{m}{n} = x_i$.

The linearized system around an equilibrium point (x_i, k), $k \in R$, has the matrix

$$A_2 = \begin{pmatrix} q + 3x_i^2 & 0 \\ n & 0 \end{pmatrix}$$

and the corresponding eigenvalues are the same as for the matrix A_1, namely $\lambda_1 = 0, \lambda_2 = q + 3x_i^2$. The equations of the phase trajectories can be obtained considering $\frac{dx}{dy} = \frac{r + qx + x^3}{m + nx}$.

Consequently $\frac{m+nx}{r+qx+x^3}dx = dy$ and $y = y(x)$ follow in every situation $D > 0$, $D < 0$, $D = 0$.

Consider now system (38). As from the second equation (39) it follows $y = -\frac{m}{s}$, system (38) can have one, three or two equilibria if $D > 0$, $D < 0$ or $D = 0$, respectively. The linearized system around an equilibrium point $(x_i, -\frac{m}{s})$, $i = 0, 1, 2, 3$, has the matrix

$$A_3 = \begin{pmatrix} q + 3x_i^2 & 0 \\ 0 & s \end{pmatrix}$$

and the corresponding eigenvalues are $\lambda_1 = s \neq 0, \lambda_2 = q + 3x_i^2$.

In order to find the type of the equilibrium points, the sign of λ_2 must be considered. Thus, if $D > 0$, $\lambda_2 = q + 3x_0^2$. Replacing the expression of x_0 given by (41), (42) or (43) into the expression of λ_2, it follows that $\lambda_2 > 0$. Consequently, if $s > 0$, then the equilibrium point $(x_0, -\frac{m}{s})$ is a repulsor and if $s < 0$, it is a saddle point.

If $D < 0$, using (44) we get $\lambda_2 = q + 3x_1^2 = q\left[1 - 4sin^2\left(\Phi + \frac{\pi}{3}\right)\right] > 0$. Consequently, the equilibrium point $(x_1, -\frac{m}{s})$ is a repulsor for $s > 0$ and a saddle for $s < 0$. Using (45), $\lambda_2 = q + 3x_2^2 = q(1 - 2sin\Phi)(1 + 2sin\Phi) < 0$ so $(x_2, -\frac{m}{s})$ is an attractor for $s < 0$ and a saddle for $s > 0$. Using (46), we get $\lambda_2 = q + 3x_3^2 = q(1 - 2sin\left(\frac{\pi}{3} - \Phi\right))(1 + 2sin\left(\frac{\pi}{3} - \Phi\right)) > 0$, so the equilibrium point $(x_3, -\frac{m}{s})$ is a repulsor for $s > 0$ and a saddle for $s < 0$.

If $D = 0$, using (47) and (48) it follows that $\lambda_2 = 0$ for the double equilibrium point and $\lambda_2 > 0$ for the simple equilibrium point. Thus, the simple equilibrium is a repulsor for $s > 0$ and a saddle for $s < 0$, while the double equilibrium point is nonhyperbolic. Its type will be deduced using the center manifold theory [5]. Using the transformation $u = x - x_i, \quad v = y + \frac{m}{s}$ where x_i is the abscissa of the double equilibrium point and taking into account that $\lambda_2 = 0$, system (38) becomes

$$\dot{u} = 3x_i u^2 + u^3, \quad \dot{v} = sv. \tag{49}$$

System (49) has the origin as an equilibrium point with the eigenvalues $\lambda_1 = s, \lambda_2 = 0$. The center manifold must be of the form

$$v = V(u) = \gamma_1 u^2 + \gamma_2 u^3 + ... \tag{50}$$

Replacing (50) into the second equation (49), we get

$$\frac{\partial V}{\partial u} \dot{u} = sV(u)$$

which is equivalent with

$$\left(2\gamma_1 u + 3\gamma_2 u^2 + ...\right)\left(3x_i u^2 + u^3\right) = s\left(\gamma_1 u^2 + \gamma_2 u^3 + ...\right)$$

It follows $\gamma_1 = \gamma_2 = ... = 0$. Consequently, the center manifold is $V(u) = 0$ and the flow on the center manifold is given by $\dot{u} = 3x_i u^2 + u^3$. As $x_i \neq 0$, it follows that the double equilibrium point is a nondegenerate saddle-node.

5 F-N system

The classical F-N system reads [1]

$$\dot{x} = c_{FN}(x + y - x^3/3), \quad \dot{y} = -(x - a_{FN} + yb_{FN})/c_{FN}$$

therefore it corresponds to the coefficients $a = 0$, $b = \frac{a_{FN}}{c_{FN}}$, $c = d = c_{FN}$, $e = -\frac{1}{c_{FN}}$, $f = -\frac{b_{FN}}{c_{FN}}$, $p = -c_{FN}$. Since $c_{FN} \neq 0$ it follows that the corresponding parameters belong to a set (manifold) of the $Aff_3(2, R)$-group. Let $N_3 = N_3(j_2p_2 + j_3 \neq 0)$. We have $N_3 = N_3' U N_3''$, where $N_3' = N_3'(j_3 \neq 0)$ and $N_3''(j_2p_2 \neq 0)$ are two disjoint sets. They are invariant with respect to the $Aff_3(2, R)$-group. On N_3' and N_3'' the system (1) has the following canonical forms

$$\dot{x} = a + cx + dy + px^3, \quad \dot{y} = b + ex + fy, \tag{51}$$

where $p \neq 0$, i.e. the given system (1),

$$\dot{x} = a + cx + px^3, \quad \dot{y} = b + ex + fy, \tag{52}$$

where $pe \neq 0$, $pf \neq 0$. Hence the classical F-N system is of the form (51). Its main characteristic is that, in general, it cannot be decoupled (i.e. it is not of separate variables). This is mainly due to the fact that the parameter d, which has a crucial role in the dynamics generated by (51), is nonzero. Since for the F-N system we have $c = d$, it follows that it never takes the forms (30)–(33), (52). This explains the big differences between the dynamics generated by the classical F-N system and the dynamics generated by (30)–(33), sketched in the previous section.

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