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Classification of quadratic systems with a symmetry center and simple infinite singular points

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Abstract. We classify the family of planar quadratic differential systems with a center of symmetry and two invariant straight lines according to the topology of their phase portraits. The case of the existence of simple infinite singular points is only considered. For each of the obtained distinct topological classes we give necessary and sufficient conditions in terms of algebraic invariants and comitants. The program was implemented for computer calculations.

Mathematics subject classification: 34C14, 34C05, 58F14.
Keywords and phrases: quadratic differential system, invariant line, phase portrait, limit cycle.

1 Introduction and the statement of main results

Consider generic quadratic systems of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dx}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y) \end{aligned} \tag{1}$$

with real homogeneous polynomials $p_i,\ q_i \in \mathbb{R}[\mathbf{a}, x, y]$ $(i = 0, 1, 2)$ of degree i in x, y.

In paper [10] the notion of a *dicritical* (not necessarily singular) point of a quadratic differential system is introduced. As particular cases, it comprises *symmetry* point of the corresponding integral curves, *dicritical nodal singular* point and *homogeneity* point (i.e., such point that system (1) becomes homogeneous after shifting the point to the origin). The class of quadratic system with homogeneity point was studied in [2, 7, 11–13, 16, 19, 21–23]. In papers [4, 20] the topological classification of system (1) having a dicritical nodal singular point is obtained. Some classes of the quadratic systems (1) possessing a symmetry point were examined in papers [3, 17, 18].

The purpose of our article is the study of quadratic system (1) with a symmetry point and two parallel invariant straight lines which can be: (a) real distinct; (b) imaginary; (c) coincided in the finite part of the phase plane; (c) coincided at infinity. For this class of system (1) all possible topological distinct phase portraits

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will be constructed and the respective necessary and sufficient conditions for their realization will be established.

We introduce the following polynomials:

$$C_i = yp_i(x,y) - xq_i(x,y) \ (i = 0,1,2), \qquad D_i = \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y} \ (i = 1,2),$$

which in fact are GL-comitants [5, 16]. To formulate the statement of the Main Theorem we shall construct T-comitants and CT-comitants (see [15] for detailed definitions) which distinguish phase portraits of the class of system (1) possessing a center of symmetry and two parallel invariant straight lines. All of them will be constructed only by using polynomials C_i and D_i via the differential operator $(f,g)^{(k)}$ called *transvectant of the index* k [8,14] which acts on $\mathbb{R}[\mathbf{a},x,y]$ as follows:

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

Here $f(x,y)$ and $g(x,y)$ are polynomials in x, y of the degree r and ρ, respectively, and $\mathbf{a} \in \mathbb{R}^{12}$ is 12-tuple of the coefficients of system (1).

First we construct the following comitants of the second degree with respect to coefficients of initial system (1):

$$\begin{array}{lll} T_1 = (C_0,C_1)^{(1)}, & T_2 = (C_0,C_2)^{(1)}, & T_3 = (C_0,D_2)^{(1)}, \\ T_4 = (C_1,C_1)^{(2)}, & T_5 = (C_1,C_2)^{(1)}, & T_6 = (C_1,C_2)^{(2)}, \\ T_7 = (C_1,D_2)^{(1)}, & T_8 = (C_2,C_2)^{(2)}, & T_9 = (C_2,D_2)^{(1)}. \end{array}$$

By using the initial T-comitants: $\tilde{A}$, $\tilde{B}$, $\tilde{C} \equiv C_2$, $\tilde{D}$, $\tilde{E}$, $\tilde{F}$, $\tilde{G} \equiv D_2$, $\tilde{H}$, $\tilde{K}$ written in tensorial form in paper [5] was constructed a minimal polynomial basis of T-comitants of system (1) up to degree 12.

We shall use here some of these T-comitants, expressed through C_i and D_j:

$$\begin{aligned} \tilde{A}(\mathbf{a}) = {} & \left(C_1, T_8 - 2T_9 + D_2^2\right)^{(2)}/144, \\ \widetilde{D}(\mathbf{a},x,y) = {} & \big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(T_7 - T_6) - (C_1,T_5)^{(1)} \\ & +6D_1(C_1D_2 - T_5) - 9D_1^2C_2\big]/36, \\ \widetilde{E}(\mathbf{a},x,y) = {} & \big[D_1(2T_9 - T_8) - 3\,(C_1,T_9)^{(1)} - D_2(3T_7 + D_1D_2)\big]/72, \\ \widetilde{F}(\mathbf{a},x,y) = {} & \big[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0\,(D_2,T_9)^{(1)} - D_2^2T_4 + 288D_1\widetilde{E} \\ & -24\big(C_2,\widetilde{D}\big)^{(2)} + 120\big(D_2,\widetilde{D}\big)^{(1)} - 36C_1\,(D_2,T_7)^{(1)} + 8D_1\,(D_2,T_5)^{(1)}\big]/144, \\ \widetilde{K}(\mathbf{a},x,y) = {} & (T_8 + 4T_9 + 4D_2^2)/72, \\ \widetilde{H}(\mathbf{a},x,y) = {} & (-T_8 + 8T_9 + 2D_2^2)/72. \end{aligned}$$

Now the needed T-comitants expressed only through the polynomials C_i $(i = 0,1,2)$

and D_j $(j = 1, 2)$ via differential operator $(*, *)^{(k)}$ can be constructed:

$$
\begin{aligned}
M(\mathbf{a}, x, y) =&\ T_8/8 \equiv \text{Hessian}(C_2)/4,\\
K(\mathbf{a}, x, y) =&\ \widetilde{K}(\mathbf{a}, x, y) \equiv \big(p_2(x, y), q_2(x, y)\big)^{(1)}/4,\\
N_1(\mathbf{a}, x, y) =&\ (T_8 - 2T_9 + D_2^2)/36,\\
N_2(\mathbf{a}, x, y) =&\ \big[D_1(2T_9 - T_8 - 3D_2^2) - 3D_2T_7 - 3\,(C_1, T_9)^{(1)}\big]/72,\\
N_5(\mathbf{a}, x, y) =&\ (T_5 - 3C_2D_1 + 2C_1D_2)/6,\\
V(\mathbf{a}, x, y) =&\ \big[4(T_2 + C_0D_2)^2 - 3(T_5 - 3C_2D_1 + 2C_1D_2)(T_1 + C_0D_1)\big]/36,\\
W_1(\mathbf{a}, x, y) =&\ 2\big(C_2, \widetilde{D}\big)^{(2)} - 7\big(D_2, N_1\big)^{(2)} + 18\widetilde{F},\\
W_2(\mathbf{a}, x, y) =&\ 15\,C_2\big[23\big((\widetilde{D}, \widetilde{D})^{(2)}, D_2\big)^{(1)} + 7\big((C_2, \widetilde{D})^{(2)}, \widetilde{D}\big)^{(2)}\big] - 11\big[\big(C_2, \widetilde{D}\big)^{(2)}\big]^2 +\\
&\ 36\widetilde{D}\big[42\big(C_2, \widetilde{F}\big)^{(2)} - 197\big(\widetilde{D}, \widetilde{K}\big)^{(2)} + 184\big(\widetilde{D}, \widetilde{H}\big)^{(2)}\big] +\\
&\ 6D_2\big[168\big(\widetilde{D}, \widetilde{F}\big)^{(1)} - 19\big((C_2, \widetilde{D})^{(2)}, \widetilde{D}\big)^{(1)}\big] +\\
&\ 288\widetilde{F}\big[2\big(C_2, \widetilde{D}\big)^{(2)} + 9\widetilde{F}\big] + 172\big(C_2, \widetilde{D}\big)^{(3)}\big(C_2, \widetilde{D}\big)^{(1)} +\\
&\ 12\big(49\widetilde{K} - 197\widetilde{H}\big)\big(\widetilde{D}, \widetilde{D}\big)^{(2)} - 194\big(C_2, \widetilde{D}\big)^{(2)}\big(D_2, \widetilde{D}\big)^{(1)},\\
W_3(\mathbf{a}, x, y) =&\ \big(\big(C_2, \widetilde{D}\big)^{(1)}, \big(C_2, \widetilde{D}\big)^{(1)}\big)^{(2)} - 6\big(C_2, \widetilde{D}\big)^{(1)}\big(C_2, \widetilde{D}\big)^{(3)},\\
\eta(\mathbf{a}) =&\ (M, M)^{(2)}/6 \equiv \text{Discrim}(C_2),\\
\mu(\mathbf{a}) =&\ -(K, K)^{(2)}/2 \equiv \text{Discrim}(K),\\
\kappa(\mathbf{a}) =&\ -(N_1, N_1)^{(2)}/8 \equiv \text{Discrim}(N_1)/4,\\
G_1(\mathbf{a}) =&\ \big(C_1, T_8 - 2T_9 + D_2^2\big)^{(1)}/144,\\
H_1(\mathbf{a}) =&\ 9\big(\big((\widetilde{D}, \widetilde{D}\big)^{(2)}, D_2\big)^{(1)}, D_2\big)^{(1)} + 270\big((\widetilde{D}, \widetilde{D})^{(2)}, (6\widetilde{K} + N_1)\big)^{(2)} +\\
&\ 576\big((\widetilde{D}, \widetilde{F})^{(2)}, D_2\big)^{(1)} + 396\big((C_2, \widetilde{D})^{(2)}, \widetilde{F}\big)^{(2)} - 86\big[\big(C_2, \widetilde{D}\big)^{(3)}\big]^2,\\
H_2(\mathbf{a}) =&\ \big(\widetilde{H}, \widetilde{K}\big)^{(2)} - 3\big(\widetilde{H}, \widetilde{H}\big)^{(2)},\\
H_3(\mathbf{a}) =&\ -6\big(\widetilde{F}, \widetilde{K}\big)^{(2)} - 4\big((\widetilde{D}, \widetilde{H})^{(2)}, D_2\big)^{(1)} - \big((\widetilde{D}, \widetilde{K})^{(2)}, D_2\big)^{(1)},\\
F_1(\mathbf{a}) =&\ 10\big[\big(C_2, \widetilde{D}\big)^{(3)}\big]^2 - 99\big(\big((\widetilde{D}, \widetilde{D}\big)^{(2)}, D_2\big)^{(1)}, D_2\big)^{(1)} - 36\big((C_2, \widetilde{D})^{(2)}, \widetilde{F}\big)^{(2)} +\\
&\ 54\big((\widetilde{D}, \widetilde{D})^{(2)}, (7\widetilde{H} - \widetilde{K})\big)^{(2)} - 288\big((\widetilde{D}, \widetilde{F})^{(2)}, D_2\big)^{(1)},\\
F_2(\mathbf{a}) =&\ \big(\widetilde{H}, \widetilde{K}\big)^{(2)} + \big(\widetilde{H}, \widetilde{H}\big)^{(2)},\\
F_3(\mathbf{a}) =&\ \big(C_2, \widetilde{D}\big)^{(3)},\\
E_1(\mathbf{a}) =&\ 4\big((\widetilde{D}, \widetilde{F})^{(2)}, D_2\big)^{(1)} + 3\big((\widetilde{D}, \widetilde{D})^{(2)}, (\widetilde{K} + 3\widetilde{H})\big)^{(2)} - 4\big((C_2, \widetilde{D})^{(2)}, \widetilde{F}\big)^{(2)},\\
E_2(\mathbf{a}) =&\ \big((\widetilde{D}, N_1)^{(2)}, D_2\big)^{(1)},\\
E_3(\mathbf{a}) =&\ \big(((\widetilde{D}, D_2)^{(2)}, D_2)^{(1)}, D_2\big)^{(1)}.
\end{aligned}
$$

In order to formulate the statement of the Main Theorem we note that the geometrical meaning of the condition $\kappa = 0$ is given by Lemma 1.

Main Theorem. *For $\kappa = 0$ the phase portraits of the non-degenerate quadratic system (1) with a point of symmetry and such that polynomial $C_2 = yp_2(x, y) - xq_2(x, y) \neq 0$ has only simple roots (i.e.,$\eta \neq 0$), are determined by the respective affine invariant conditions given in Table 1. Here by r_i (respectively, c_i) the real (respectively, imaginary) singular point of multiplicity i is denoted.*

Table 1

<table>
<tr><th>Infinite singular points</th><th>Condi-tions</th><th>Finite singular points</th><th>Conditions</th><th>Phase portrait</th><th colspan="2">Additional conditions for determining phase portraits</th></tr>
<tr><td rowspan="18">$r_1r_1r_1$</td><td rowspan="18">$\eta > 0$</td><td rowspan="5">$r_1r_1r_1r_1$</td><td rowspan="5">$W_2 > 0,$ $W_1 > 0$</td><td rowspan="3">Figure 1</td><td colspan="2">$N_1 \geq 0$</td></tr>
<tr><td rowspan="2">$N_1 < 0$</td><td>$W_3 < 0$</td></tr>
<tr><td>$W_3 \geq 0, E_1 > 0$</td></tr>
<tr><td>Figure 2</td><td colspan="2">$N_1 < 0, W_3 = 0, E_1 < 0$</td></tr>
<tr><td>Figure 3</td><td colspan="2">$N_1 < 0, W_3 > 0, E_1 < 0$</td></tr>
<tr><td rowspan="4">$c_1c_1c_1c_1$</td><td rowspan="4">$W_2 < 0$ or $W_2 > 0$ & $W_1 < 0$</td><td rowspan="3">Figure 4</td><td colspan="2">$N_1 = 0$</td></tr>
<tr><td rowspan="2">$N_1 \neq 0$</td><td>$W_3 \neq 0$</td></tr>
<tr><td>$W_3 = 0, N_1E_1 > 0$</td></tr>
<tr><td>Figure 5</td><td colspan="2">$W_3 = 0, N_1E_1 < 0$</td></tr>
<tr><td rowspan="4">$r_2\,r_2$</td><td rowspan="4">$W_2 = 0$ $W_1 > 0$</td><td rowspan="2">Figure 6</td><td colspan="2">$N_1 = 0$</td></tr>
<tr><td colspan="2">$N_1 \neq 0,\ E_2 = 0$</td></tr>
<tr><td>Figure 7</td><td colspan="2">$E_2 \neq 0,\ N_1 > 0$</td></tr>
<tr><td>Figure 8</td><td colspan="2">$E_2 \neq 0,\ N_1 < 0$</td></tr>
<tr><td>$c_2\,c_2$</td><td>$W_2 = 0, W_1 < 0$</td><td>Figure 4</td><td colspan="2">–</td></tr>
<tr><td>r_4</td><td>$\mu \neq 0, W_1 = 0$</td><td>Figure 9</td><td colspan="2">–</td></tr>
<tr><td rowspan="3">–</td><td rowspan="3">$\mu = 0, V \neq 0$</td><td rowspan="2">Figure 4</td><td colspan="2">$W_3 \neq 0$</td></tr>
<tr><td colspan="2">$W_3 = 0, E_3 < 0$</td></tr>
<tr><td>Figure 5</td><td colspan="2">$W_3 = 0, E_3 > 0$</td></tr>
<tr><td rowspan="11">$r_1c_1c_1$</td><td rowspan="11">$\eta < 0$</td><td>$r_1r_1r_1r_1$</td><td>$\mu > 0, W_2 > 0$ $W_1 > 0$</td><td>Figure 10</td><td colspan="2">–</td></tr>
<tr><td rowspan="3">$r_1r_1c_1c_1$</td><td rowspan="3">$\mu < 0, W_2 \neq 0$</td><td>Figure 15</td><td colspan="2">$H_1 \neq 0$</td></tr>
<tr><td>Figure 16</td><td colspan="2">$H_1 = 0,\ H_3 > 0$</td></tr>
<tr><td>Figure 17</td><td colspan="2">$H_1 = 0,\ H_3 < 0$</td></tr>
<tr><td>$c_1c_1c_1c_1$</td><td>$W_2 < 0$ or $\mu > 0, W_2 > 0$ & $W_1 < 0$</td><td>Figure 11</td><td colspan="2">–</td></tr>
<tr><td rowspan="2">$r_2\,r_2$</td><td rowspan="2">$W_2 = 0, W_1 > 0$</td><td>Figure 12</td><td colspan="2">$E_2 = 0$</td></tr>
<tr><td>Figure 13</td><td colspan="2">$E_2 \neq 0$</td></tr>
<tr><td>$c_2\,c_2$</td><td>$W_2 = 0, W_1 < 0$</td><td>Figure 11</td><td colspan="2">–</td></tr>
<tr><td rowspan="2">r_4</td><td rowspan="2">$\mu \neq 0, W_1 = 0$</td><td>Figure 14</td><td colspan="2">$\mu > 0$</td></tr>
<tr><td>Figure 18</td><td colspan="2">$\mu < 0$</td></tr>
</table>

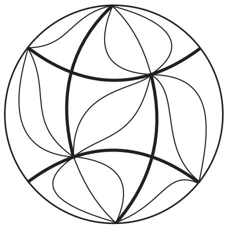


Figure 1

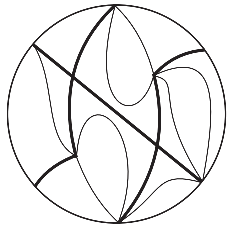
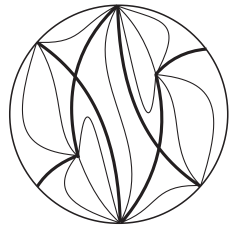


Figure 2



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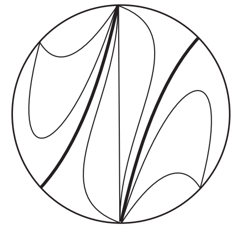


Figure 4

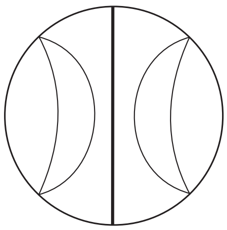
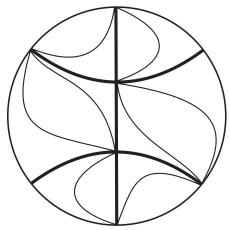


Figure 5



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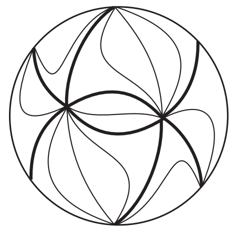

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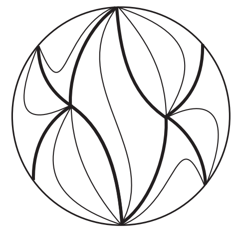

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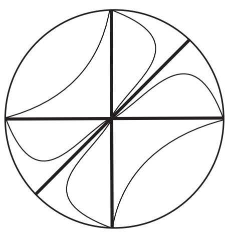

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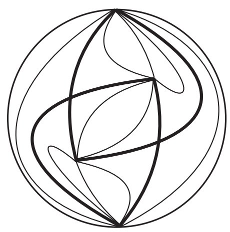
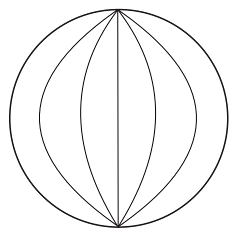


Figure 10



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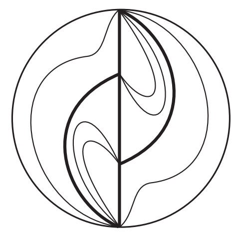

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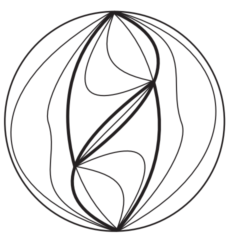

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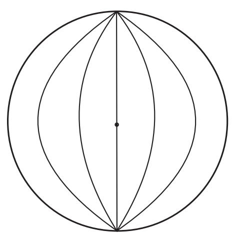

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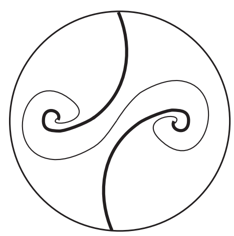

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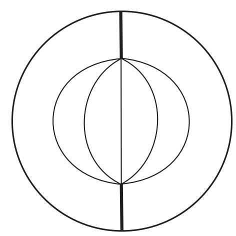

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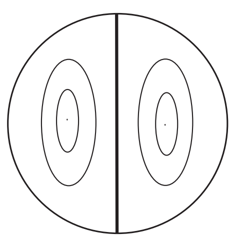

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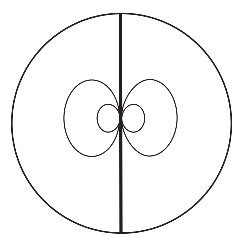

Figure 18

2 Some preliminary results

Proposition 1. [10] *System 1 has a single symmetry point if and only if either* $N_1(\mathbf{a}, x, y) \neq 0$ *and* $G_1(\mathbf{a}) = N_2(\mathbf{a}, x, y) = 0$ *or* $N_1(\mathbf{a}, x, y) = N_3(\mathbf{a}, x, y) = 0,$

$N_4(\mathbf{a}, x, y) \neq 0$; *and it has an infinite number of such points if and only if* $N_1(\mathbf{a}, x, y) = N_3(\mathbf{a}, x, y) = N_4(\mathbf{a}, x, y) = N_5(\mathbf{a}, x, y) = 0$.

Proposition 2. [16] *The number of distinct roots (real and imaginary) of the polynomial* $C_2 = yp_2(x, y) - xq_2(x, y) \not\equiv 0$ *is determined by the following conditions:*

- *3 real for* $\eta > 0$;
- *1 real and 2 imaginary for* $\eta < 0$;
- *2 real (one double and one simple) for* $\eta = 0,\ M \neq 0$;
- *1 real (triple) for* $\eta = M = 0$.

Proposition 3. [9] *The number and the types of the finite singular points of the non-homogeneous system (1) with a point of symmetry are determined in Table 1. The notations 'sdl', 'nod', 'sdl-nod', 'foc' and 'cnt' are used to denote saddle, node, saddle-node, focus, and center, respectively, and by* $(\mathfrak{A}_1)$ *we denote the following set of conditions:*

$$F_1 = F_3 = 0,\ F_2 \geq 0. \qquad (\mathfrak{A}_1)$$

The geometrical meaning of the condition $\kappa = 0$ is given by the next lemma.

Lemma 1. *Assume that for the quadratic system (1) with a point of symmetry and* $C_2 \neq 0$ *the condition* $\kappa = 0$ *holds. Then this system possesses two parallel invariant straight lines which can be: (a) real distinct; (b) imaginary; (c) coincided in the finite part of the phase plane; (c) coincided at infinity.*

Proof: We shall consider all the cases given by Proposition 2.

Case $\eta > 0$. Applying an affine transformation system (1) with a point of symmetry can be brought [16] to the canonical form

$$\dot{x} = a + gx^2 + (h-1)xy, \quad \dot{y} = b + (g-1)xy + hy^2.$$

For this system we have $\kappa = (1-g)(h-1)(g+h)/8$. So, the condition $\kappa = 0$ yields $(g-1)(h-1)(g+h) = 0$ and without loss of generality we may assume $h = 1$. Indeed, if $g = 1$ (respectively, $g + h = 0$) we can apply the linear transformation $x = y_1$, $y = x_1$ (respectively, $x = -y_1$, $y = x_1 - y_1$). Thus, $h = 1$ and we obtain the system

$$\dot{x} = a + gx^2, \quad \dot{y} = b + (g-1)xy + y^2 \qquad (2)$$

which, evidently, possesses two parallel invariant straight lines: $gx^2 + a = 0$ ($g^2 + a^2 \neq 0$). Clearly we obtain the case (a) (respectively, (b); (c); (d)) indicated in the statement of Lemma 1 when $ag < 0$ (respectively, $ag > 0$; $a = 0$; $g = 0$).

Case $\eta < 0$. According to [16] via an affine transformation system (1) can be brought to the canonical form

$$\dot{x} = a + gx^2 + (h+1)xy, \quad \dot{y} = b - x^2 + gxy + hy^2. \qquad (3)$$

Table 1

Singular points	*Affine invariant conditions*	*Characters*	*Additional conditions for determining characters*	
$r_1r_1r_1r_1$	$\mu>0, W_1>0, W_2>0$	*sdl, sdl, nod, nod*	$H_1\geq 0$	
		sdl, sdl, foc, foc	$H_1<0,\ \neg(\mathfrak{A}_1)$	
		sdl, sdl, foc, cnt	$H_1<0,\quad (\mathfrak{A}_1)$	
$r_1r_1c_1c_1$	$\mu<0,\ W_2\neq 0$	*sdl, sdl*	$K<0$	
		nod, nod	$K>0$	$H_1<0,\ H_3>0$
				$H_1=0, H_2<0, H_3>0$
				$H_1=0,\ H_2\geq 0$
				$H_1>0,\ H_2>0$
		foc, foc	$K>0,\ \neg(\mathfrak{A}_1)$	$H_1<0,\ H_3<0$
				$H_1=0, H_2<0, H_3<0$
				$H_1>0,\ H_2<0$
		cnt, cnt	$K>0,\ (\mathfrak{A}_1)$	$H_1<0,\ H_3<0$
				$H_1=0, H_2<0, H_3<0$
				$H_1>0,\ H_2<0$
$c_1c_1c_1c_1$	$\mu>0$ *and* $W_2<0$ *or* $W_2>0,\ W_1\leq 0$	—	—	
r_2r_2	$\mu>0, W_1>0, W_2=0$	*sdl-nod, sdl-nod*	$F_2\neq 0$	
		cusp, cusp	$F_2=0$	
c_2c_2	$\mu>0, W_1<0, W_2=0$	—	—	
r_4	$\mu\neq 0, W_1=0, W_2=0$	—	*Homogeneous system ([16])*	
r_1r_1	$\mu=0,\ W_1>0$	*sdl, sdl*	$K<0$	
		nod, nod	$K>0$	$H_1\geq 0$
		foc, foc	$K>0$	$H_1<0,\ \neg(\mathfrak{A}_1)$
		cnt, cnt	$K>0$	$H_1<0,\ (\mathfrak{A}_1)$
c_1c_1	$\mu=0,\ W_1<0$	—	—	
—	$\mu=0, W_1=0, V\neq 0$	—	*There are no singular points*	
—	$\mu=0, W_1=0, V=0$	—	*System is degenerate*	

For system (3) we have $\kappa=(h+1)\big[(h-1)^2+g^2\big]/8$, and the condition $\kappa=0$ yields two subcases: $h+1=0$ and $h-1=g=0$.

Subcase $h=-1$. The system (3) becomes $\dot{x}=a+gx^2,\quad \dot{y}=b-x^2+gxy-y^2$, which has the parallel lines $a+gx^2=0$.

Subcase $h-1=g=0$. We obtain the system $\dot{x}=a+2xy,\quad \dot{y}=b-x^2+y^2$, which possesses the following two couples of imaginary invariant straight lines:

$$(x-iy)^2=b+ia,\quad (x+iy)^2=b-ia.$$

Case $\eta=0,\ M\neq 0$**.** System (1) by means of an affine transformation can be brought [16] to the canonical form

$$\dot{x} = a + gx^2 + hxy, \quad \dot{y} = b + (g-1)xy + hy^2. \tag{4}$$

For this system we have $\kappa = h^2(1-g)/8$ and, hence, the condition $\kappa = 0$ implies either $g = 1$ or $h = 0$.

Subcase $g = 1$. Evidently, in this case system (4) possesses two parallel invariant straight lines $hy^2 + b = 0$ types of which are governed by parameters h and b.

Subcase $h = 0$. The system (4) becomes $\dot{x} = a + gx^2, \quad \dot{y} = b + (g-1)xy$, and again possesses the invariant straight lines $gx^2 + a = 0$.

Case $M = 0, \ C_2 \neq 0$. Via an affine transformation system (1) with a point of symmetry can be brought [16] to the canonical form

$$\dot{x} = a + gx^2 + hxy, \quad \dot{y} = b - x^2 + gxy + hy^2.$$

For this system we have $\kappa = h^3/8$ and the condition $\kappa = 0$ yields $h = 0$. This leads to the system $\dot{x} = a + gx^2, \quad \dot{y} = b - x^2 + gxy$, which possesses two parallel invariant straight lines $gx^2 + a = 0$. Lemma 1 is proved.

3 The proof of the Main Theorem

In what follows we assume that the condition $\kappa = 0$ is fulfilled.

3.1 Systems with 3 real roots of C_2

According to Proposition 2 the condition $\eta > 0$ holds. It was shown in the proof of Lemma 1 that in this case the quadratic system can be brought to the canonical form

$$\dot{x} = a + gx^2, \quad \dot{y} = b + (g-1)xy + y^2 \tag{5}$$

for which we have:

$$C_2 \equiv yp_2(x,y) - xq_2(x,y) = xy(x-y), \quad \mu = g^2.$$

Then we conclude that the intersection point of the line $x = 0$ (respectively, $y = 0$; $y = x$) with Poincaré's circumference is a real infinite singular point of system (5), which we will denote by $\widetilde{N}_1(0,1,0)$ (respectively, $\widetilde{N}_2(1,0,0)$; $\widetilde{N}_3(1,1,0)$). Since the conditions $\eta > 0$ and $\mu \neq 0$ are fulfilled in accordance with the paper [15] at infinity there exist one saddle and two nodes on the Poincaré circumference. In what follows we need to know where exactly the saddle is placed. So, by using the transformation $x = v/z, \ y = 1/x, \ dt = zd\tau$ system (5) will be brought to the system

$$\frac{dv}{d\tau} = -v + v^2 + az^2 - bvz^2, \quad \frac{dz}{d\tau} = -z + (1-g)vz - bz^3, \tag{6}$$

whereas applying the transformation $x = 1/z, \ y = u/z, \ dt = zd\tau$ we obtain the system

$$\frac{du}{d\tau} = -u + u^2 + bz^2 - auz^2, \quad \frac{dz}{d\tau} = -gz - az^3. \tag{7}$$

Clearly, the point $\widetilde{N}_1(0,1,0)$ corresponds to the singular point $(0,0)$ of system (6) and the point $\widetilde{N}_2(1,0,0)$ (respectively, $\widetilde{N}_3(1,1,0)$) corresponds to the singular point $(0,0)$ (respectively, $(1,0)$) of system (7).

Considering the eigenvalues of the corresponding linear matrix for each of these singular points we obtain, respectively:

$$\widetilde{N}_1(0,1,0): \ \lambda_1\lambda_2 = 1; \quad \widetilde{N}_2(1,0,0): \ \lambda_1\lambda_2 = g; \quad \widetilde{N}_3(0,1,0): \ \lambda_1\lambda_2 = -g.$$

Hence, we have the next affirmation:

Remark 1. *For system (5) with $\mu \neq 0$ the infinite singular point $N_1(0,1,0)$ is a node and the point $N_2(1,0,0)$ (respectively, $N_3(1,1,0)$) is a node (respectively, a saddle) for $g > 0$ and a saddle (respectively, a node) for $g < 0$.*

Let us emphasize some useful geometrical proprieties of system (5).

Remark 2. *For $g^2 - 1 = 0$ system (5) possesses two couples of parallel invariant straight lines. Moreover, one couple of parallel lines is directed to the node $N_1(0,1,0)$ and the second one is directed to the node $N_2(1,0,0)$ (respectively, node $N_3(1,1,0)$) for $g = 1$ (respectively, $g = -1$).*

Remark 3. *For $b = 0$ (respectively, $b = a$) system (5) possesses one invariant straight line which passes through the infinite singular point $N_2(1,0,0)$ (respectively, $N_3(1,1,0)$).*

For system (5) one can calculate

$$\begin{array}{l} W_1 = -24g\left[a(g-1)^2 + 2bg\right]x^2 - 48ag\,(g-1)\,xy - 48\,agy^2, \\ W_2 = 2^7 3^3 ag^2\left[a(g-1)^2 + 4bg\right]\left[(g-1)\,x + 2\,y\right]^2 x^2, \\ H_1 = 2^5 3^4\left[a(g-1)(3g-1) - 4bg\right]^2, \quad F_2 = -4g^2, \\ Discrim(W_1) = -2^8 3^2 ag^2\left[a(g-1)^2 + 4bg\right], \quad \mu = g^2. \end{array} \tag{8}$$

Case $W_2 > 0$. Then $a\left[a(g-1)^2 + 4bg\right] > 0$, and we obtain $Discrim(W_1) < 0$. Hence, the quadratic form $W_1(x,y)$ became sign definite. Moreover, by (8) we obtain $\mathrm{sign}(W_1) = -\mathrm{sign}(ag)$. Since $ag \neq 0$ by applying the transformation

$$x = \alpha x_1, \quad y = \alpha y_1, \quad t = \alpha^{-1}t_1, \quad (\alpha = \sqrt{|ag^{-1}|}), \tag{9}$$

system (5) can be brought to the following canonical form (we keep the previous notations):

$$\dot{x} = g\left(x^2 + Sign(ag)\right), \quad \dot{y} = b + (g-1)xy + y^2. \tag{10}$$

<u>*Subcase*</u> $W_1 > 0$. Then $ag < 0$ and system (10) becomes

$$\dot{x} = g\left(x^2 - 1\right), \quad \dot{y} = b + (g-1)xy + y^2. \tag{11}$$

This system possesses two parallel real invariant straight lines: $x = \pm 1$. Since by (8) we have $H_1 \geq 0$, according to Proposition 3 for $W_2 > 0$, $W_1 > 0$ and $H_1 \geq 0$

system (5) has 4 real singular points placed on the invariant straight lines $x = \pm 1$ and namely, two saddles and two nodes: $M_1^{\pm}\left(1, y_1^{\pm}\right)$, $\quad M_2^{\pm}\left(-1, y_2^{\pm}\right)$, where

$$y_1^{\pm} = \frac{1-g \pm \sqrt{\Delta}}{2}, \quad y_2^{\pm} = \frac{g-1 \pm \sqrt{\Delta}}{2}, \quad \Delta = (g-1)^2 - 4b.$$

We note that $\Delta > 0$ because of $W_2 > 0$. The symmetry of the vector field of system (11) implies the symmetry of the point M_1^+ with M_2^- as well as the symmetry of the point M_1^- with M_2^+. Thus, it is sufficient to determine only the types of the points $M_1^{\pm}$. It is not difficult to calculate the corresponding eigenvalues and to find out for each point:

$$M_1^+ : \quad \lambda_1\lambda_2 = 2g\sqrt{\Delta}; \qquad M_1^- : \quad \lambda_1\lambda_2 = -2g\sqrt{\Delta}.$$

1) $g < 0$. Then the singular point M_1^+ (respectively, M_1^-) is a saddle (respectively, a node), and $y_1^+ > y_1^-$. Taking into account the coordinates of the singular points we observe that the straight line which connects the saddles M_1^+ and M_2^- will be

$$y = K_l x, \qquad K_l = \frac{1-g+\sqrt{\Delta}}{2} = y_1^+ > 0.$$

Remark 4. *It is known ([24], Lemma 11.4) that if the line passing through two singular points of quadratic system is not an invariant straight line, then it must be a line without contact except singular points.*

In order to determine the position of the separatrices of the saddle M_1^+ with respect to the line $y = y_1^+ x$, we shall determine the direction of the proper vectors of the linear matrix corresponding to this singular point. So, besides the evident direction $x = 1$ we obtain the direction: $y = K_s x, \quad K_s = (1-g)y_1^+/(\sqrt{\Delta} - 2g) > 0$. It is not difficult to determine that for $g < 0$ the following relations hold:

$$\begin{array}{llll} K_s < K_l & iff \;\; g \le -1 \; or \; -1 < g < 0, \; b < -g & \Leftrightarrow & Figure\ 1; \\ K_s = K_l & iff \;\; -1 < g < 0, \; b = -g & \Leftrightarrow & Figure\ 2; \\ K_s > K_l & iff \;\; -1 < g < 0, \; b > -g & \Leftrightarrow & Figure\ 3. \end{array} \tag{12}$$

We observe that for $b = -g$ we obtain $y_1^+ = 1$ and then the line $y = x$ becomes invariant straight line of system (11) which connects two saddles M_1^+ and M_2^-. Hence we obtain Figure 2.

Taking into consideration Remark 4 we conclude that inside the domain bounded by the invariant straight lines $x = \pm 1$ the separatrix will connect the saddle M_1^+ with the node M_1^+ for $K_s < K_l$ (Figure 1) and with the infinite node $N_1(0,1,0)$ for $K_s > K_l$ (Figure 3).

2) For $g > 0$ we obtain that the singular point M_1^+ (respectively, M_1^-) is a node (respectively, a saddle), and $y_1^+ > y_1^-$. In the same manner as above we can examine the directions of the separatrices for the saddle M_1^-. And it is not too hard to determine that for $(g-1)^2 - 2b > 0$ and $g > 0$ the corresponding phase portraits for the canonical system (11) will be realized if and only if the following conditions are fulfilled, respectively:

$$\begin{array}{lll} Figure\ 1 & iff & g \geq 1 \ or \ 0 < g < 1 \ and \ b < 0; \\ Figure\ 2 & iff & 0 < g < 1,\ b = 0; \\ Figure\ 3 & iff & 0 < g < 1,\ b > 0. \end{array} \tag{13}$$

It remains to find out the corresponding affine invariant conditions. For the system (11) we have

$$\begin{array}{l} E_1 = 384(2b+g)g^2(g^2-1), \quad N_1 = (g^2-1)x^2/4, \\ W_3 = -648b(b+g)(g^2-1)^2x^4. \end{array} \tag{14}$$

Taking into consideration (12), (13) and (14) it is not too difficult to obtain the following correspondence between Figures 1-3 and respective affine invariant conditions:

$$\begin{array}{lll} Figure\ 1 & iff & N_1 \geq 0 \ or \ N_1 < 0 \ and \ either \ W_3 < 0 \ or \ W_3 \geq 0 \ and \ E_1 > 0; \\ Figure\ 2 & iff & N_1 < 0,\ W_3 = 0,\ E_1 < 0; \\ Figure\ 3 & iff & N_1 < 0,\ W_3 > 0,\ E_1 < 0. \end{array}$$

Subcase $W_1 < 0$. Then $ag > 0$ and system (10) becomes

$$\dot{x} = g\left(x^2+1\right), \quad \dot{y} = b + (g-1)xy + y^2. \tag{15}$$

This system possesses two parallel imaginary invariant straight lines: $x = \pm i$ and it has no real singular points. For system (15) we have

$$\begin{array}{l} W_3 = 648b(g-b)(g^2-1)^2x^4, \\ E_1 = 384(g-2b)g^2(g^2-1), \\ N_1 = (g^2-1)x^2/4. \end{array} \tag{16}$$

1) We assume that the condition $N_1 \neq 0$ holds. By Remark 3 system (15) has one real invariant line for $b(b-g)$=0. Moreover, considering Remark 1 we obtain that this line will be a separatrix of infinite saddle if either $g < 0$ and $b = 0$ or $g > 0$ and $b = g$. By $N_1 \neq 0$ from (16) we obtain Figure 4 if either $W_3 \neq 0$ or $W_3 = 0$ and $N_1E_1 > 0$ and we obtain Figure 5 for $W_3 = 0$ and $N_1E_1 < 0$.

2) If $N_1 = 0$ then $g^2 - 1 = 0$. Since system (15) has a center of symmetry that a separatrix connection can be only if this separatrix is an invariant straight line. So, by Remark 2 we conclude that for $N_1 = 0$ the phase portrait of system (15) is given by Figure 4.

Case $W_2 < 0$. According to Proposition 3 system (5) has not real singular points and by (8) the condition $a\left[a(g-1)^2 + 4bg\right] < 0$ holds. Then system (5) has 2 parallel invariant straight lines $a + gx^2 = 0$ which connect two infinite nodes. So, we obtain the phase portrait given by Figure 5 (respectively, Figure 4) if there exists (respectively, does not exist) a separatrix connection of the infinite saddles. As it was mentioned above since system (5) has a center of symmetry then a separatrix connection can be only if this separatrix is an invariant straight line. For this system we have

$$\begin{array}{l} W_3 = 648b(a-b)(g^2-1)^2x^4, \quad N_1 = (g^2-1)x^2/4, \\ E_1 = 384ag(a-2b)(g^2-1), \end{array} \tag{17}$$

1) If $N_1 = 0$ then $g^2 - 1 = 0$ and by Remark 2 we obtain that there can not exist a separatrix connection. Therefore we get Figure 4.

2) We assume that the condition $N_1 \neq 0$ holds. According to Remark 1 for $g < 0$ (respectively, $g > 0$) the infinite saddle is located at the point $N_2(1,0,0)$ (respectively, $N_3(1,1,0)$). Therefore, by Remark 2 we obtain a separatrix connection if and only if either $b = 0$ and $g < 0$ or $b = a$ and $g > 0$. Taking into account (17) we conclude that the phase portrait of system (5) is given by Figure 5 for $W_3 = 0$ and $N_1E_1 < 0$ and it is given by Figure 5 if either $W_3 \neq 0$ or $W_3 = 0$ and $N_1E_1 > 0$.

Case $W_2 = 0$**.** Then $a\left[a(g-1)^2 + 4bg\right] = 0$, and according to (8) we obtain $Discrim(W_1) = 0$. Therefore, $W_1(x, y)$ became sign definite quadratic form and we shall consider three subcases: $W_1 > 0$, $W_1 < 0$ and $W_1 = 0$.

<u>*Subcase*</u> $W_1 > 0$. From (8) we obtain $g \neq 0$ and then $\mu > 0$ and $F_2 \neq 0$. By Proposition 3 system (5) has 2 double singular points which are saddle-nodes. For this system we have $N_1 = (g^2-1)x^2/4, \quad E_2 = -8ag(g^2-1)$.

1) If $N_1 = 0$ then $g^2 - 1 = 0$ and without loss of generality we can assume $g = 1$, otherwise the transformation $x_1 = -x$, $y_1 = y - x$ and $g \to -g$ which keeps canonical system (5) can be applied. Then we obtain the system

$$\dot{x} = a + x^2, \quad \dot{y} = b + y^2 \tag{18}$$

for which $W_2 = 2^{11}3^3abx^2y^2, \quad W_1 = -48(bx^2 + ay^2)$. Therefore, the conditions $W_2 = 0$ and $W_1 \neq 0$ yield $ab = 0$ and $a^2 + b^2 \neq 0$. We can assume $b = 0$ (via changing $x \leftrightarrow y$) and from $W_1 > 0$ we get $a < 0$. Thus, system (5) possesses 3 invariant lines $x = \pm\sqrt{-a}$ and $y = 0$ as well as 2 saddle-nodes $(\pm\sqrt{-a}, 0)$. So, we get the phase portrait given by Figure 6.

2) We assume now that the condition $N_1 \neq 0$ holds. Then $g^2 - 1 \neq 0$ and we shall consider two subcases: $E_2 = 0$ and $E_2 \neq 0$.

a) If $E_2 = 0$ then by $W_1N_1 \neq 0$ we obtain $a = 0$ (then $W_2 = 0$) and from (8) the condition $W_1 > 0$ yields $b < 0$. Then the saddle-nodes $(0, \pm\sqrt{-b})$ of system (5) are placed on the double invariant straight line $x = 0$. So, we get again Figure 6.

b) For $E_2 \neq 0$ we have $a \neq 0$ and the condition $W_2 = 0$ yields $a(g-1)^2 + 4bg = 0$. Since $g - 1 \neq 0$ we can substitute for b a new parameter u by setting $b = u(g-1)^2$ and then we have $a = -4gu$. Thus, we obtain the system

$$\dot{x} = -4gu + gx^2, \quad \dot{y} = u(g-1)^2 + (g-1)xy + y^2 \tag{19}$$

for which we have: $W_2 = 0, \quad W_1 = 48ug^2\left[(g-1)x + 2y\right]^2$. Hence, the condition $W_1 > 0$ yields $u > 0$. System (19) has 2 real invariant straight lines $x = \pm 2\sqrt{u}$ and two singular points which are saddle-nodes: $M_{1,2}(\pm 2\sqrt{u},\ \pm(1-g)\sqrt{u})$. We shall examine more detailed the singular point M_1. After the transformation

$$x_1 = x + \frac{4g}{(g-1)^2}y + \frac{2(g+1)\sqrt{u}}{g-1}, \quad y_1 = x - 2\sqrt{u} \quad \text{and} \quad t_1 = 4g\sqrt{u}\,t \tag{20}$$

which removes this point to the origin of coordinates, we obtain the standard [1] canonical system

$$\dot{x}_1 = \frac{1}{16g^2\sqrt{u}}[(g-1)x_1 + (g+1)y_1]^2, \quad \dot{y} = y_1 + \frac{1}{4\sqrt{u}}y_1^2. \tag{21}$$

Following [1] we obtain $\psi(x) = \tilde{\Delta}_2\, x^2 + \ldots = \dfrac{(g-1)^2}{16g^2\sqrt{u}}\, x^2 + \ldots$, and, hence, the semi-axis $y_1 = 0, x_1 < 0$ is one of the separatrices of the saddle-node $M_1(0,0)$ and other two separatrices are tangent to the axis $x_1 = 0$ at this point.

On the other hand the second saddle-node $M_2(x_0,\, y_0)$ of system (21) with coordinates $x_0 = 4(g+1)\sqrt{u}(g-1)$, $y_0 = -4\sqrt{u}$ is placed on the invariant line $y_1 = -4\sqrt{u}$ and $x_0 > 0$ for $g^2 - 1 > 0$ and $x_0 < 0$ for $g^2 - 1 < 0$. We observe that the transformation (20) removed infinite singular point as following:

$$\widetilde{N}_1(0,1,0) \to \widehat{N}_1(1,0,0);\;\; \widetilde{N}_2(1,0,0) \to \widehat{N}_2(1,1,0);\;\; \widetilde{N}_3(1,1,0) \to \widehat{N}_3\Big(1, \frac{(g-1)^2}{(g+1)^2}, 0\Big).$$

Thus, taking into consideration Remark 1 and the fact that according to Remark 4 M_0M_1 is a segment without contact, we obtain Figure 7 for $N_1 > 0$ and Figure 8 for $N_1 < 0$.

Subcase $W_1 < 0$. From (8) we obtain $g \neq 0$ and then $\mu > 0$. Then by Proposition 3 system (5) has 2 double imaginary singular points. Since system (5) has a center of symmetry then a separatrix connection can be only if this separatrix is an invariant straight line. We claim that this system can not possesses an invariant straight line as a separatrix. Indeed, by Remark 3 the condition $b(b-a) = 0$ must be satisfied. By (8) the condition $W_2 = 0$ yields $a\left[a(g-1)^2 + 4bg\right] = 0$. Then $a \neq 0$, otherwise for $a = 0$ the condition $b(b-a) = 0$ contradicts $W_1 = -48bg^2x^2 < 0$. Therefore, we obtain $a(g-1)^2 + 4bg = 0$.

If $b = 0$ we obtain $g = 1$ and by Remark 3 the invariant straight line $y = 0$ of system (5) connect two nodes. For $b = a$ we have $(g-1)^2 + 4g = (g+1)^2 = 0$, i.e. $g = -1$ and we again obtain that the invariant line $y = x$ connects two nodes. The claim is proved. Consequently, we get Figure 4.

Subcase $W_1 = 0$. From (8) we obtain $g\left[a(g-1)^2 + 2bg\right] = ag(g-1) = ag = 0$.

1) Assume $\mu \neq 0$. Then by (8) we have $g \neq 0$ and, hence, $a = b = 0$. Consequently, system (5) becomes quadratic homogeneous system, which according to Remark 1 has at infinity two nodes and one saddle. So, we get Figure 9.

1) For $\mu = 0$ from (8) we obtain $g = 0$ and system (5) becomes

$$\dot{x} = a, \quad \dot{y} = b - xy + y^2 \tag{22}$$

for which we have:

$$\begin{aligned} &\mu = W_1 = W_2 = 0, \quad V = a^2y^2(x-y)^2 \neq 0,\\ &W_3 = 648b(a-b)x^4, \quad E_3 = 24(2b-a). \end{aligned} \tag{23}$$

Taking into consideration systems (6) and (7) (for $g = 0$) we conclude that the singular point $N_1(0,1,0)$ is a node, and according to [1] the triple singular point

$N_2(1,0,0)$ (respectively, $N_3(1,1,0)$) is a node (respectively, a saddle) for $a>0$ and a saddle (respectively, a node) for $a<0$.

By Remark 3 we conclude that system (22) has an invariant straight line which connects two infinite saddles if and only if either $b=0$ and $a<0$ or $b=a$ and $a>0$. So, considering (23) we obtain Figure 5 if $W_3=0$, $E_3>0$ and Figure 4 if either $W_3\neq 0$ or $W_3=0$ and $E_3<0$.

3.2 Systems with 1 real and 2 imaginary roots of C_2

According to Proposition 2 the condition $\eta>0$ holds and according to [16] the system can be brought to the canonical form

$$\dot{x}=a+gx^2+(h+1)xy,\quad \dot{y}=b-x^2+gxy+hy^2. \tag{24}$$

For this system we have

$$\begin{aligned}&\kappa=(h+1)\Big[(h-1)^2+g^2\Big]/8,\quad C_2\equiv yp_2(x,y)-xq_2(x,y)=x(x^2+y^2),\\ &N_1=[(g^2-2h+2)x^2+2g(h+1)xy+(h^2-1)y^2]/4,\end{aligned} \tag{25}$$

and, hence, $N_1(0,1,0)$ is a real infinite singular point of this system. On the other hand the condition $\kappa=0$ yields two cases: $h+1=0$ and $h-1=g=0$ which are equivalent to $N_1\neq 0$ and $N_1=0$, respectively.

Case $N_1\neq 0$**.** Then $h=-1$ and we obtain the system

$$\dot{x}=a+gx^2,\quad \dot{y}=b-x^2+gxy-y^2, \tag{26}$$

for which

$$\begin{aligned}&W_1=-24g\left[a(g^2-2)-2bg\right]x^2+48ag^2xy-48\,agy^2,\\ &W_2=2^73^3ag^2\left[a(g^2-4)-4bg\right][g\,x-2\,y]^2\,x^2,\\ &H_1=2^53^4\left[3ag^2+4bg+4a\right]^2,\qquad F_2=-4g^2,\\ &Discrim(W_1)=-2^83^2ag^2\left[a(g^2-4)-4bg\right],\quad \mu=g^2.\end{aligned} \tag{27}$$

If $\mu\neq 0$ then from (27) it follows $\mu>0$ and since $\eta<0$ the singular point $N_1(0,1,0)$ is a node [15].

<u>*Subcase*</u> $W_2>0$. Then $a\left[a(g^2-4)-4bg\right]>0$ and by (27) we obtain $Discrim(W_1)<0$, and, hence, $\text{sign}(W_1)=-\text{sign}(ag)$. Since $ag\neq 0$ by applying the transformation (9) we get the system:

$$\dot{x}=g\left(x^2+Sign(ag)\right),\quad \dot{y}=b-x^2+gxy-y^2. \tag{28}$$

1) If $W_1>0$ then $ag<0$ and system (28) becomes

$$\dot{x}=g\left(x^2-1\right),\quad \dot{y}=b-x^2+gxy-y^2. \tag{29}$$

This system possesses two parallel real invariant straight lines: $x=\pm 1$. Since by (27) we have $H_1\geq 0$ according to Proposition 3 for $W_2>0$, $W_1>0$ and $H_1\geq 0$

system (26) has 4 real singular points located on the invariant straight lines $x = \pm 1$ and namely, two saddles and two nodes: $M_1^{\pm}\left(1, y_1^{\pm}\right)$, $M_2^{\pm}\left(-1, y_2^{\pm}\right)$, where

$$y_1^{\pm} = (g \pm \sqrt{\Delta})/2, \quad y_2^{\pm} = (-g \pm \sqrt{\Delta})/2, \quad \Delta = g^2 + 4b - 4 > 0.$$

For the singular points $M_1^{\pm}$ we have $\lambda_1 \lambda_2 = \mp 2g\sqrt{\Delta}$. We can assume $g > 0$ via the transformation $y \leftrightarrow -y$ and $t \leftrightarrow -t$. In this case M_1^+ is a saddle and M_1^- is a node and $y_1^+ > y_1^-$. Taking into account the coordinates of the singular points we observe that the straight line $y = y_1^+ x$ connects the saddles M_1^+ and M_2^-.

On the other hand the directions of the separatrices of the saddle M_1^+ are $x = 1$ and $y = K_s x$, where $K_s = \dfrac{g y_1^+ - 2}{g + 2y_1^+}$. Therefore, $K_s - y_1^+ = -\left[\left(y_1^+\right)^2 + 2\right]/(g + 2y_1^+) < 0$ by $g > 0$. Thus, the located inside the domain $-1 < x < 1$ separatrix of the saddle M_1^+ by Remark 4 must connect this saddle with the node M_2^-. So, we get Figure 10.

2) Condition $W_1 < 0$ implies $ag > 0$ and system (28) has no real singular points. Taking into account the infinite node we obtain Figure 11.

Subcase $W_2 < 0$. According to Proposition 3 system (28) has no real singular points and we again get Figure 11.

Subcase $W_2 = 0$. Then $a\left[a(g^2 - 4) - 4bg\right] = 0$ and by (27) we obtain $Discrim(W_1) = 0$. Therefore, $W_1(x, y)$ became sign definite quadratic form and we shall consider three subcases: $W_1 > 0$, $W_1 < 0$ and $W_1 = 0$.

1) If $W_1 > 0$ then from (27) we obtain $g \neq 0$ and then $\mu > 0$ and $F_2 \neq 0$. By Proposition 3 system (26) has 2 double singular points which are saddle-nodes. For this system we have $E_2 = -8ag(g^2 + 4)$.

a) If $E_2 = 0$ then $a = 0$ and the saddle-nodes are located on the invariant straight line $x = 0$ of system (26). So, we obtain Figure 12.

b) For $E_2 \neq 0$ we have $ag \neq 0$ and the condition $W_1 > 0$ by (27) yields $ag < 0$. Then we obtain system (29) for which the condition $W_2 = 0$ yields $g^2 + 4b - 4 = 0$. Therefore, we get the system

$$\dot{x} = g\left(x^2 - 1\right), \quad \dot{y} = 1 - g^2/4 - x^2 + gxy - y^2 \tag{30}$$

with two real invariant straight lines $x = \pm 1$ and two saddle-nodes $M_1(1, g/2)$ and $M_2(-1, -g/2)$. We can assume $g > 0$, otherwise the substitution $y \leftrightarrow -y$, $t \leftrightarrow -t$ and $g \leftrightarrow -g$ can by applied. On the line $y = gx/2$ which connects singular points M_1 and M_2 we have $dy/dx = (g^2 - 4)/(4g)$ and by $g > 0$ we obtain $(g^2 - 4)/(4g) - g/2 = -(g^2 + 4)/(4g) < 0$. Consequently, we get Figure 13.

2) For $W_1 < 0$ according to Proposition 3 system (28) has no real singular points and we obtain Figure 11.

3) Assume $W_1 = 0$. From (27) we obtain $g\left[a(g^2 - 2) - 2bg\right] = ag = 0$.

a) For $\mu \neq 0$ we have $g \neq 0$ and, hence, $a = b = 0$. Consequently, system (26) becomes a quadratic homogeneous system which has a unique real infinite singular point (a node). So, we get Figure 14.

b) If $\mu = 0$ then from (27) we obtain $g = 0$ and system (26) becomes

$$\dot{x} = a, \quad \dot{y} = b - x^2 - y^2$$

which has not finite singular points and has one real simple infinite point (a node). Therefore we obtain Figure 11.

Case $N_1 = 0$**.** Then by (25) we have $h - 1 = g = 0$ and we obtain the system

$$\dot{x} = a + 2xy, \quad \dot{y} = b - x^2 + y^2, \tag{31}$$

for which

$$\begin{array}{l} W_1 = -96(bx^2 - 2axy + by^2), \quad W_2 = 2^{11}3^3\left(a^2 + b^2\right)\left(x^2 + y^2\right)^2, \\ \mu = -4, \quad K = x^2 + y^2, \quad F_1 = 2^{13}3^4a^2 = H_1, \\ F_2 = 16 = -H_2, \quad H_3 = -2^9 b, \quad F_3 = -192a. \end{array} \tag{32}$$

Since $\mu < 0$ and $\eta < 0$ the singular point $N_1(0, 1, 0)$ is a saddle (see,[15]).

Subcase $W_2 \neq 0$. Then according to Proposition 3 system (31) has 2 real and 2 imaginary singular points.

1) If $H_1 \neq 0$ by (32) we have $H_1 > 0$ and since $K > 0$ and $H_2 < 0$ according to Proposition 3 the real points of system (31) are foci. We claim that this system can not possess limit cycles. Indeed, condition $H_1 \neq 0$ yields $a \neq 0$ and via the transformation

$$x_1 = sign(a)\,|a|^{-1/2}x, \quad y_1 = |a|^{-1/2}y, \quad t_1 = |a|^{1/2}x, \quad b = c^2 - \frac{1}{4c^2}$$

system (31) becomes

$$\dot{x} = 1 + 2xy, \quad \dot{y} = c^2 - \frac{1}{4c^2} - x^2 + y^2. \tag{33}$$

This system possesses the following two couples of parallel imaginary invariant straight lines:

$$x - iy = \pm\frac{2c^2 + i}{2c}, \quad x + iy = \pm\frac{2c^2 - i}{2c}.$$

Following [6] we construct the first integral of system (33) in the complex form:

$$\big(x - iy - c - \frac{i}{2c}\big)^{i-2c^2}\big(x - iy + c + \frac{i}{2c}\big)^{2c^2-i}\big(x + iy - c + \frac{i}{2c}\big)^{-i-2c^2}\big(x + iy + c - \frac{i}{2c}\big)^{i+2c^2}.$$

Then the corresponding real first integral of system (33) can be constructed:

$$\exp\Big[-2arctg\Big(\frac{4cx + 8c^3y}{1+4c^4-4c^2(x^2+y^2)}\Big)\Big]\left(\frac{1 + 4c^4 + 8c^3x - 4cy + 4c^2(x^2 + y^2)}{1 + 4c^4 - 8c^3x + 4cy + 4c^2(x^2 + y^2)}\right)^{2c^2}.$$

Since the curve $1 + 4c^4 - 4c^2(x^2 + y^2) = 0$ is not a particular solution of system (33) and the identity

$$1 + 4c^2 - 8c^3x + 4cy + 4c^2(x^2 + y^2) = 4c^2\big(x - iy - c - \frac{i}{2c}\big)\big(x + iy - c + \frac{i}{2c}\big)$$

holds, we conclude that our claim is proved.

Taking into account that the line $x = 0$ is not an invariant straight line of system (33) we obtain Figure 15.

2) Assume $H_1 = 0$. Then $a = 0$ and by Proposition 3 system (31) has two nodes located on the invariant straight line $x = 0$ for $H_3 > 0$ (Figure 16) and it has two centers for $H_3 < 0$ (Figure 17).

Subcase $W_2 = 0$. By (32) we have $a = b = 0$ and system (31) becomes a homogeneous system with one real invariant straight line which is a separatrix of the saddle $N_1(0, 1, 0)$. Therefore, we obtain Figure 18.

In order to obtain the respective to the case $\eta < 0$ conditions from Table 1 the following Remark has to be taking into account:

Remark 5. *For system (26) with $\kappa = 0$ from (27) and (32) we obtain:*
- *condition $\mu < 0$ is equivalent to $N_1 = 0$;*
- *conditions $W_2 = 0$, $W_1 \neq 0$ implies $\mu > 0$;*
- *condition $W_1 = 0$ implies $W_2 = 0$.*

The Main Theorem is proved.

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